Math. Proc. Camb. Phil. Soc. (2010), **148**, 87 © *Cambridge Philosophical Society* 2009
doi:10.1017/S030500410999020X
First published online 29 October 2009

On the compact real form of the Lie algebra $\mathfrak{g}_2$

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(*Received* 20 *March* 2009; *revised* 15 *May* 2009)

Abstract

We give an elementary construction of the compact real form of the Lie algebra $\mathfrak{g}_2$. This construction exhibits the group $2^{3\cdot}L_3(2)$ as a group of automorphisms. We also show that there is a unique 14-dimensional real Lie algebra invariant under the action of this group.

1. *Introduction*

It is known that every simple complex Lie algebra has a (unique) compact real form, defined by the property that the Killing form is negative definite. For example, the compact real form of $\mathfrak{a}_1$ is just the well-known 3-dimensional vector cross product given by

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$$

and images under the symmetry rotating **i**, **j** and **k**. In terms of the Chevalley basis e, f, h we may take $2\mathbf{i} = \sqrt{-1}h$, $2\mathbf{j} = e + f$, and $2\mathbf{k} = \sqrt{-1}(e - f)$. Indeed, Jacobson [**3**, p.149] gives the basis

$$\left\{\sqrt{-1}h_j,\, e_\alpha + e_{-\alpha},\, \sqrt{-1}(e_\alpha - e_{-\alpha})\right\}$$

for the compact real form in terms of the Chevalley basis for the split real form of any simple complex Lie algebra.

However, this basis does not necessarily exhibit as much symmetry as is possible: all that is guaranteed is that a subgroup $2^r.W$ of the stabilizer of a Cartan subalgebra remains, where W denotes the Weyl group. As the above example shows, we can sometimes do better than this. A particularly desirable aim is to write the algebra as a direct sum of mutually orthogonal Cartan subalgebras and, if possible, to have a symmetry group permuting these Cartan subalgebras transitively. The book by Kostrikin and Tiep [**4**] is devoted to such decompositions of simple Lie algebras, and contains a wealth of interesting information, including of course explicit constructions of them. In particular, on pp. 104–106 and 203–207, two different constructions of the compact real form of $\mathfrak{g}_2$ are described. The first is given explicitly as an algebra of derivations of a Cayley algebra, but the seven mutually orthogonal Cartan subalgebras are not obviously permuted transitively. In the second case, a more complicated construction using a loop produces a transitive group $2^{3\cdot}L_3(2)$, acting irreducibly. In this paper I give a more elementary construction of the compact real form of $\mathfrak{g}_2$, exhibiting this irreducible subgroup $2^{3\cdot}L_3(2)$ of the automorphism group.

2. *Representations of* $2^{3\cdot}L_3(2)$

The group $2^{3\cdot}L_3(2)$ is a non-split extension of an elementary abelian group of order 8 by the general linear group $GL_3(2)$ acting on it. This group acts on the Lie algebra of type G_2 by permuting seven mutually orthogonal Cartan subalgebras, and the stabilizer of one of these is a group $2^{3\cdot}S_4 \cong 4^2.(2\times S_3)$. The latter description shows that the induced action on the Cartan subalgebra is the full Weyl group, $W(G_2)\cong 2\times S_3$, and that the representation of $2^{3\cdot}L_3(2)$ on the Lie algebra is induced from the natural 2-dimensional representation of the Weyl group. (In fact, there are two such representations, but they are interchanged by the outer automorphism of $2^{3\cdot}L_3(2)$, so it does not matter which one we pick.)

To construct this representation, we first take the 7-dimensional representation of $2^{3\cdot}L_3(2)$ generated with respect to a basis $\{i_t \mid t\in\mathbb{F}_7\}$ of a (real) vector space V by the maps $\alpha\colon t\mapsto t+1$, $\beta\colon t\mapsto 2t$ and the involution $\gamma=(i_2,-i_2)(i_4,-i_4)(i_3,i_5)(i_6,i_0)$. It is clear that these maps preserve the set of lines $\{t,t+1,t+3\}$ of the projective plane of order 2, and it is easy to check that the kernel of the action is precisely the group 2^3 of sign-changes on the complements of the lines. Thus they generate a group of shape $2^{3\cdot}L_3(2)$.

Now construct the exterior square of this representation, on the basis

$$\{u_t := i_{t+2}\wedge i_{t+6},\ v_t := i_{t+4}\wedge i_{t+5},\ w_t := i_{t+1}\wedge i_{t+3}\}.$$

The three generators given above act as follows (where x stands for an arbitrary one of u, v, w, and basis vectors not mentioned are fixed):

$$\begin{aligned}
\alpha &: x_t \longmapsto x_{t+1}\\
\beta &: u_t \longmapsto v_{2t} \longmapsto w_{4t} \longmapsto u_t\\
\gamma &: u_1 \longleftrightarrow v_1,\ x_2 \longleftrightarrow -x_2,\ u_4 \longleftrightarrow -w_4,\\
&\quad u_3 \longleftrightarrow v_5,\ v_3 \longleftrightarrow w_5,\ w_3 \longleftrightarrow u_5,\\
&\quad u_6 \longleftrightarrow w_0,\ v_6 \longleftrightarrow v_0,\ w_6 \longleftrightarrow u_0.
\end{aligned}$$

There is a submodule spanned by the $u_t+v_t+w_t$, isomorphic to the original 7-dimensional module spanned by the i_t. Factoring this out leaves the 14-dimensional module we require. We use the same notation, now with the understanding that $u_t+v_t+w_t=0$. Write L_t for the 2-space spanned by u_t, v_t, and w_t, and let $L=\bigoplus_{t\in\mathbb{F}_7} L_t$.

3. *Products*

Next consider what products on this 14-space are invariant under the group $2^{3\cdot}L_3(2)$. Invariance under the normal 2^3 implies that the product of a vector in L_r with one in L_s lies in L_t, where $\{r,s,t\}$ is a line in the projective plane. Invariance under α means we only need to consider the products on one line, say the line $\{1,2,4\}$. Modulo the sign-changes on the L_t, the stabilizer of this line is a group S_4, generated by β, γ, and $\gamma^{\alpha^{-1}}$.

Invariance under γ implies that $[u_1+v_1,x_2]$ is a scalar multiple of u_4+w_4, and that $[u_1-v_1,x_2]$ is a scalar multiple of u_4-w_4. So we may assume that $[u_1,v_2]=\lambda w_4+\mu u_4$ and $[v_1,v_2]=\mu w_4+\lambda u_4$, and scale so that $\lambda+\mu=1$. Applying the symmetry γ^β gives the values of $[u_1,w_2]$ and $[v_1,w_2]$, and the multiplication table can be filled in using the relations $u_t+v_t+w_t=0$:

	u_2	v_2	w_2
u_1	v_4	$\lambda w_4+\mu u_4$	$\lambda u_4+\mu w_4$
v_1	v_4	$\lambda u_4+\mu w_4$	$\lambda w_4+\mu u_4$
w_1	$-2v_4$	v_4	v_4

This gives us a 1-parameter family of (non-associative) algebras. The symmetry $\gamma^{\beta^{-1}}$ shows that the multiplication in every such algebra is anti-commutative, and the symmetry γ shows that the multiplication is zero on L_2, and therefore on each L_t. The full multiplication table may be constructed from the extract given above using the anti-commutativity and the symmetries α and β. It is straightforward, if a little tedious, to verify that all these algebras are invariant under $2^{3\cdot}L_3(2)$.

LEMMA 1. *The multiplication given above satisfies the Jacobi identity if and only if* $\mu = 0$ *(equivalently,* $\lambda = 1$*).*

Proof. The Jacobi identity $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ holds trivially if the three vectors a, b, c are respectively in L_r, L_s and L_t where $\{r, s, t\}$ is a line. Now the symmetries show that it is necessary and sufficient for the Jacobi identity to hold for the triples (u_1, v_1, v_2), (u_3, u_6, v_5), and (u_3, v_6, u_5). In the first case we have

$$\begin{aligned}[[u_1, v_1], v_2] + [[v_1, v_2], u_1] + [[v_2, u_1], v_1] &= 0 + [\lambda u_4 + \mu w_4, u_1] - [\lambda w_4 + \mu u_4, v_1]\\ &= 0\end{aligned}$$

since $[u_4, u_1] = [w_4, v_1]$ and $[w_4, u_1] = [u_4, v_1]$. So this case of the identity holds whatever the values of λ and μ. In the second case we have

$$\begin{aligned}[[u_3, u_6], v_5] + [[u_6, v_5], u_3] + [[v_5, u_3], u_6] &= -[\lambda w_4 + \mu v_4, v_5] - [v_1, u_3] + 2[w_2, u_6]\\ &= -\lambda v_0 - \lambda\mu u_0 - \mu^2 w_0 - v_0 + 2v_0\\ &= \mu(v_0 - \lambda u_0 - \mu w_0).\end{aligned}$$

But the expression in brackets is never zero, so $\mu = 0$, $\lambda = 1$, and the multiplication table is determined. We can now verify that

$$\begin{aligned}[[u_3, v_6], u_5] + [[v_6, u_5], u_3] + [[u_5, u_3], v_6] &= -[u_4, u_5] - [w_1, u_3] - [w_2, v_6]\\ &= -v_0 - u_0 - w_0 = 0\end{aligned}$$

which is the final case of the Jacobi identity.

COROLLARY 1. *Up to scalar multiplication, there is a unique* 14*-dimensional Lie algebra invariant under the given action of* $2^{3\cdot}L_3(2)$.

Observe that, in order to remember the definition of the algebra, it is sufficient, given the anti-symmetry and the rule that $u_t + v_t + w_t = 0$, to remember the table

	v_2	w_2
u_1	w_4	u_4
v_1	u_4	w_4

and the maps $\alpha\colon x_t \mapsto x_{t+1}$ and $\beta\colon u_t \mapsto v_{2t} \mapsto w_{4t} \mapsto u_t$. At the referee's request, however, I include the full multiplication table in the appendix.

4. *Identification of the Lie algebra*

As usual, we denote by $\operatorname{ad} x$ the linear map $y \mapsto [x, y]$ on L, and define the Killing form by $(x, y) := \operatorname{Tr}(\operatorname{ad} x . \operatorname{ad} y)$. It is obvious that the Killing form is a symmetric bilinear form.

LEMMA 2. *The Killing form on L is negative definite.*

Proof. First observe that if $x \in L_0$ then $\mathrm{ad}\, x$ maps L_t into $L_{\pi(t)}$ where π is the permutation $(1,3)(2,6)(4,5)$. Similarly if $y \in L_1$ then $\mathrm{ad}\, y$ effects the permutation $(2,4)(3,0)(5,6)$. Therefore $\mathrm{ad}\, x.\mathrm{ad}\, y$ maps every L_t into a different L_t, so has trace 0. Hence the L_t are mutually orthogonal with respect to the Killing form.

Now we can calculate the Killing form on L_0 using the following two rows of the multiplication table of the algebra:

	v_1	w_1	w_2	u_2	w_3	u_3	u_4	v_4	v_5	w_5	u_6	v_6
u_0	w_3	u_3	w_6	w_6	$-v_1$	$-w_1$	w_5	v_5	$-v_4$	$-u_4$	$-v_2$	$-v_2$
v_0	u_3	w_3	u_6	v_6	$-w_1$	$-v_1$	u_5	u_5	$-w_4$	$-w_4$	$-w_2$	$-u_2$

We find that $\mathrm{Tr}\,(\mathrm{ad}\, u_0.\mathrm{ad}\, u_0) = -16$ and $\mathrm{Tr}\,(\mathrm{ad}\, u_0.\mathrm{ad}\, v_0) = 8$, so that the Killing form is negative definite on each L_t.

THEOREM 1. *The Lie algebra L is the compact real form of $\mathfrak{g}_2$.*

Proof. We have shown that the Killing form is non-singular, which implies that the Lie algebra L is semisimple. It is easy to see that each L_t is a Cartan subalgebra, so that L has rank 2. The classification of complex semisimple Lie algebras shows immediately that L is of type G_2. Since the Killing form is negative definite, L is the compact real form.

Alternatively, a proof from first principles can be obtained by explicitly diagonalising $\mathrm{ad}\, u_0$ and $\mathrm{ad}\, v_0$ simultaneously (over $\mathbb{C}$). Their simultaneous eigenspaces are the root spaces, and one recovers the standard construction of the split real form.

5. *Further remarks*

Since each L_t is a Cartan subalgebra, it contains a natural copy of the G_2 root system. Our spanning vectors have been chosen so that the short roots in L_t are (up to a suitable scaling factor) $\pm u_t$, $\pm v_t$ and $\pm w_t$. The long roots, similarly, are $\pm(u_t - v_t)$, $\pm(v_t - w_t)$ and $\pm(w_t - u_t)$.

The multiplication table given above can be used to define a (non-associative) algebra on a 21-space over any field of characteristic not 2, invariant under the given action of $2^{3\cdot}L_3(2)$. The 7-dimensional subspace spanned by the $u_t + v_t + w_t$ is a subalgebra (indeed, an ideal) isomorphic to the algebra of pure imaginary octonions. This subalgebra satisfies the Jacobi identity if and only if the field has characteristic 3. Moreover, its orthogonal complement is closed under multiplication if and only if the field has characteristic 3. In this case, the multiplication on the orthogonal complement also satisfies the Jacobi identity, and we recover the fact that the pure imaginary octonion algebra is a subalgebra (and an ideal) of the Lie algebra of type $\mathfrak{g}_2$ in characteristic 3. These ideas are exploited further in [**6**] where an elementary construction of the finite simple Ree groups ${}^2G_2(3^{2n+1})$ is given.

It would be interesting to find similar constructions for the other exceptional Lie algebras. In type F_4 we can use a group $3^3{:}L_3(3)$ permuting 13 mutually orthogonal Cartan subalgebras 2-transitively. Details are given in [**7**] but they are not as nice as in type G_2. The subgroup of the Weyl group which acts on one Cartan subalgebra is $(3 \times 2^{\cdot}A_4){:}2$, and corresponds to a description of F_4 as a 2-dimensional lattice over $\mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/3}$. This is closely related to the description of F_4 in terms of the Hurwitz ring of quaternions $\mathbb{Z}[i, \frac{1}{2}(1+i+j+k)]$.

It should be possible to extend this to type E_6, where again there are 13 mutually orthogonal Cartan subalgebras. In type E_7 there are 19 mutually orthogonal Cartan subalgebras, but there seems to be no interesting group permuting these.

Type E_8 is particularly interesting as there are two completely different groups available each of which permutes a set of 31 mutually orthogonal Cartan subalgebras 2-transitively. One of these is $5^3{:}L_3(5)$ (see [**1**, **2**]), and the other is $2^{5+10\cdot}L_5(2)$ (see [**5**]). In the first case there is a group $(5 \times 2^{\cdot}A_5).4$ acting on one Cartan subalgebra, giving it the structure of a 2-dimensional lattice over $\mathbb{Z}[e^{2\pi i/5}]$. This is closely related to the well-known description of E_8 in terms of Hamilton's ring of icosians. In the second case, the group acting on one Cartan subalgebra is $2^7 A_8$, and it should be possible to use the Coxeter–Dickson ring of integral octonions to describe the Lie algebra.

6. *Appendix*

	u_0	v_0	w_0	u_1	v_1	w_1	u_2	v_2	w_2	u_3	v_3	w_3	u_4	v_4	w_4	u_5	v_5	w_5	u_6	v_6	w_6
u_0				v_3	w_3	u_3	w_6	$-2w_6$	w_6	$-w_1$	$-u_1$	$-v_1$	w_5	v_5	u_5	$-w_4$	$-v_4$	$-u_4$	$-v_2$	$-v_2$	$2v_2$
v_0				v_3	u_3	w_3	v_6	w_6	u_6	$-v_1$	$-u_1$	$-w_1$	u_5	u_5	$-2u_5$	$2w_4$	$-w_4$	$-w_4$	$-w_2$	$-u_2$	$-v_2$
w_0				$-2v_3$	v_3	v_3	u_6	w_6	v_6	$-u_1$	$2u_1$	$-u_1$	v_5	w_5	u_5	$-w_4$	$-u_4$	$-v_4$	$-u_2$	$-w_2$	$-v_2$
u_1	$-v_3$	$-v_3$	$2v_3$				v_4	w_4	u_4	w_0	$-2w_0$	w_0	$-w_2$	$-u_2$	$-v_2$	w_6	v_6	u_6	$-w_5$	$-v_5$	$-u_5$
v_1	$-w_3$	$-u_3$	$-v_3$				v_4	u_4	w_4	v_0	w_0	u_0	$-v_2$	$-u_2$	$-w_2$	u_6	u_6	$-2u_6$	$2w_5$	$-w_5$	$-w_5$
w_1	$-u_3$	$-w_3$	$-v_3$				$-2v_4$	v_4	v_4	u_0	w_0	v_0	$-u_2$	$2u_2$	$-u_2$	v_6	w_6	u_6	$-w_5$	$-u_5$	$-v_5$
u_2	$-w_6$	$-v_6$	$-u_6$	$-v_4$	$-v_4$	$2v_4$				v_5	w_5	u_5	w_1	$-2w_1$	w_1	$-w_3$	$-u_3$	$-v_3$	w_0	v_0	u_0
v_2	$2w_6$	$-w_6$	$-w_6$	$-w_4$	$-u_4$	$-v_4$				v_5	u_5	w_5	v_1	w_1	u_1	$-v_3$	$-u_3$	$-w_3$	u_0	u_0	$-2u_0$
w_2	$-w_6$	$-u_6$	$-v_6$	$-u_4$	$-w_4$	$-v_4$				$-2v_5$	v_5	v_5	u_1	w_1	v_1	$-u_3$	$2u_3$	$-u_3$	v_0	w_0	u_0
u_3	w_1	v_1	u_1	$-w_0$	$-v_0$	$-u_0$	$-v_5$	$-v_5$	$2v_5$				v_6	w_6	u_6	w_2	$-2w_2$	w_2	$-w_4$	$-u_4$	$-v_4$
v_3	u_1	u_1	$-2u_1$	$2w_0$	$-w_0$	$-w_0$	$-w_5$	$-u_5$	$-v_5$				v_6	u_6	w_6	v_2	w_2	u_2	$-v_4$	$-u_4$	$-w_4$
w_3	v_1	w_1	u_1	$-w_0$	$-u_0$	$-v_0$	$-u_5$	$-w_5$	$-v_5$				$-2v_6$	v_6	v_6	u_2	w_2	v_2	$-u_4$	$2u_4$	$-u_4$
u_4	$-w_5$	$-u_5$	$-v_5$	w_2	v_2	u_2	$-w_1$	$-v_1$	$-u_1$	$-v_6$	$-v_6$	$2v_6$				v_0	w_0	u_0	w_3	$-2w_3$	w_3
v_4	$-v_5$	$-u_5$	$-w_5$	u_2	u_2	$-2u_2$	$2w_1$	$-w_1$	$-w_1$	$-w_6$	$-u_6$	$-v_6$				v_0	u_0	w_0	v_3	w_3	u_3
w_4	$-u_5$	$2u_5$	$-u_5$	v_2	w_2	u_2	$-w_1$	$-u_1$	$-v_1$	$-u_6$	$-w_6$	$-v_6$				$-2v_0$	v_0	v_0	u_3	w_3	v_3
u_5	w_4	$-2w_4$	w_4	$-w_6$	$-u_6$	$-v_6$	w_3	v_3	u_3	$-w_2$	$-v_2$	$-u_2$	$-v_0$	$-v_0$	$2v_0$				v_1	w_1	u_1
v_5	v_4	w_4	u_4	$-v_6$	$-u_6$	$-w_6$	u_3	u_3	$-2u_3$	$2w_2$	$-w_2$	$-w_2$	$-w_0$	$-u_0$	$-v_0$				v_1	u_1	w_1
w_5	u_4	w_4	v_4	$-u_6$	$2u_6$	$-u_6$	v_3	w_3	u_3	$-w_2$	$-u_2$	$-v_2$	$-u_0$	$-w_0$	$-v_0$				$-2v_1$	v_1	v_1
u_6	v_2	w_2	u_2	w_5	$-2w_5$	w_5	$-w_0$	$-u_0$	$-v_0$	w_4	v_4	u_4	$-w_3$	$-v_3$	$-u_3$	$-v_1$	$-v_1$	$2v_1$			
v_6	v_2	u_2	w_2	v_5	w_5	u_5	$-v_0$	$-u_0$	$-w_0$	u_4	u_4	$-2u_4$	$2w_3$	$-w_3$	$-w_3$	$-w_1$	$-u_1$	$-v_1$			
w_6	$-2v_2$	v_2	v_2	u_5	w_5	v_5	$-u_0$	$2u_0$	$-u_0$	v_4	w_4	u_4	$-w_3$	$-u_3$	$-v_3$	$-u_1$	$-w_1$	$-v_1$			

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