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A new approach to the Suzuki groups

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There are many constructions of the Suzuki groups in the literature (see for example Suzuki's original paper [**3**], as well as [**1**, **2**, **4**]), and one needs to make a strong case to justify publishing another. Yet I believe the construction below is sufficiently new and sufficiently elementary that in time it will come to be regarded as the standard construction.

First we set up the symplectic geometry. Let $F=\mathbb{F}_q$ be the field of order $q=2^{2n+1}$, and for $i=1,2$ define a 2-space $V_i=\langle e_i, e_{-i}\rangle$ with symplectic form $f_i(e_i, e_{-i})=1$. Let $V=V_1\perp V_2$ be the standard symplectic 4-space with symplectic form $f=f_1\perp f_2$. Next we want to define a commutative product $u\bullet v$ for all $u, v\in V$ which satisfy $f(u,v)=0$. To do this we first define it for all u and v and then restrict to this subset. Define $\bullet$ on the standard basis by

$$\begin{aligned} e_1\bullet e_2 &= e_2\\ e_{-1}\bullet e_2 &= e_1\\ e_1\bullet e_{-2} &= e_{-1}\\ e_{-1}\bullet e_{-2} &= e_{-2}\\ e_i\bullet e_{\pm i} &= 0 \end{aligned}$$

and extend to the whole of V by the following twisted linearity formula:

$$\left(\sum_i \lambda_i e_i\right)\bullet\left(\sum_j \mu_j e_j\right)=\sum_{i,j}(\lambda_i\mu_j)^{2^n}(e_i\bullet e_j).$$

It is then easy to see that $\bullet$ satisfies the following:

(i) $u\bullet v=v\bullet u$,
(ii) $u\bullet(v+w)=u\bullet v+u\bullet w$,
(iii) $u\bullet(\lambda v)=\lambda^{2^n}(u\bullet v)$.

It also satisfies $v\bullet v=0$, since in the expansion of this product the cross terms cancel out, and the diagonal terms are by definition zero.

Now define $G=G(q)$, where $q=2^{2n+1}$, to be the subgroup of all elements g of the symplectic group preserving f, which also satisfy $u^g\bullet v^g=u\bullet v$ for all u, v such that $f(u,v)=0$. Then in fact $G(q)$ is isomorphic to the Suzuki group $\mathrm{Sz}(q)$, and the ovoid on which it acts consists of the points $\langle v\rangle$ such that $v=v\bullet w$ for some w. (An ovoid is just a set of q^2+1 points no two of which are collinear. See [**4**] for a discussion of this family of ovoids.)

To prove these facts is quite straightforward. First observe that the map $r\colon e_i\mapsto e_{-i}$ preserves f and $\bullet$ so lies in G. Next consider linear maps of the form $e_i\mapsto\lambda_i e_i$. To preserve f, such a map must satisfy $\lambda_{-i}=\lambda_i{}^{-1}$. To preserve $\bullet$ also, it must satisfy $(\lambda_1 e_1)\bullet(\lambda_2 e_2)=\lambda_2 e_2$, that is $(\lambda_1\lambda_2)^{2^n}=\lambda_2$, so $\lambda_1\lambda_2=(\lambda_2)^{2^{n+1}}$ and therefore $\lambda_1=(\lambda_2)^{2^{n+1}-1}$, which can also be written $\lambda_2=(\lambda_1)^{2^{n+1}+1}$. Conversely, if these equations hold then $(\lambda_{-1}e_{-1})\bullet(\lambda_2e_2)=(\lambda_1^{-1}\lambda_2)^{2^n}(e_{-1}\bullet e_2)=\lambda_1e_1$, so $\bullet$ is preserved. Indeed, if we define $h(\lambda)$ to be such a diagonal element, with $\lambda_1=\lambda$, $\lambda_2=\lambda^{2^{n+1}+1}$ and $\lambda_{-i}=(\lambda_i)^{-1}$, then the $h(\lambda)$ form a maximal torus $H\cong C_{q-1}$, normalised by the Weyl group $W\cong C_2$ generated by r. We have shown that H is exactly the set of diagonal matrices contained in G.

Next we prove invariance under a root element x:

$$\begin{aligned}
e_{-2} &\mapsto e_{-2}\\
e_{-1} &\mapsto e_{-1}+e_{-2}\\
e_1 &\mapsto e_1+e_{-1}\\
e_2 &\mapsto e_2+e_1+e_{-1}+e_{-2}.
\end{aligned}$$

Checking the symplectic form is easy, as

$$\begin{aligned}
&f(e_{-2},e_{-1}+e_{-2}) = f(e_{-2},e_1+e_{-1})=0\\
&f(e_{-2},e_2+e_1+e_{-1}+e_{-2}) = 1\\
&f(e_{-1}+e_{-2},e_1+e_{-1}) = 1\\
&f(e_{-1}+e_{-2},e_2+e_1+e_{-1}+e_{-2}) = 0\\
&f(e_1+e_{-1},e_2+e_1+e_{-1}+e_{-2}) = 0.
\end{aligned}$$

In order to check the product $\bullet$, it is useful to think of $\bullet$ as corresponding to an additive (but not linear) map π from a subspace of the exterior square $V\wedge V$ to V. The symplectic form f corresponds to a linear map ϕ: $V\wedge V\to V$, and π is defined on the kernel of ϕ by $\pi(u\wedge v)=u\bullet v$, and extending additively. In combination with the rule for scalar multiples, this shows that in order to prove π is invariant under a group element g, it suffices to check it on a basis for ker ϕ.

Therefore to check that the bullet product is invariant under x it is sufficient to check it on the basis vectors for V, so we calculate

$$\begin{aligned}
e_{-2}\bullet(e_{-1}+e_{-2}) &= e_{-2}\\
e_{-2}\bullet(e_1+e_{-1}) &= e_{-1}+e_{-2}\\
(e_{-1}+e_{-2})\bullet(e_2+e_1+e_{-1}+e_{-2}) &= e_{-1}\bullet(e_2+e_{-2})+e_{-2}\bullet(e_1+e_{-1})\\
&= e_1+e_{-1}\\
(e_1+e_{-1})\bullet(e_2+e_1+e_{-1}+e_{-2}) &= (e_1+e_{-1})\bullet(e_2+e_{-2})\\
&= e_2+e_1+e_{-1}+e_{-2}\\
(e_{-1}+e_{-2})\bullet(e_1+e_{-1}) &= e_{-1}+e_{-2}\\
e_{-2}\bullet(e_2+e_1+e_{-1}+e_{-2}) &= e_{-1}+e_{-2}.
\end{aligned}$$

The last two lines of this calculation show that $\bullet$ defined on the whole space is *not* invariant under x. However, their sum shows that π defined on ker ϕ *is* invariant, which is what is required.

Thus we have proved that the standard generators r, $h(\lambda)$ and x of the Suzuki groups are contained in G.

To prove the converse, and to derive the standard properties of the Suzuki groups, consider the *points* $\langle v\rangle$ defined by the property that $v=v\bullet w$ for some w. First observe that the bullet product is *graded* in the sense that $e_i\bullet e_j=e_{g(i+j)}$, where g is the function mapping $-3,-1,0,1,3$ to $-2,-1,0,1,2$ respectively, and e_0 is interpreted as 0. This means that if we define the *degree* $d(v)$ of a vector $v=\sum_i\lambda_ie_i$ to be the largest i such that $\lambda_i\neq 0$, then the degree of $v\bullet w$ is $g(d(v)+d(w))$. From the definition of $\bullet$ it is immediate that if $\langle v\rangle$

is a point then the degree of v is either -2 or 2. In the former case $\langle v\rangle$ is the point $\langle e_{-2}\rangle$, so consider the latter case.

Conjugating the root element x by a suitable diagonal element gives a map of the form $e_2 \mapsto e_2 + \alpha e_1 +$ (lower terms) for arbitrary α, so we can assume that v has no term in e_1. Similarly, x squares to a map which takes $e_2 \mapsto e_2 + e_{-1} + e_{-2}$, and conjugating this by a suitable diagonal element gives a map which allows us to remove the term in e_{-1} from v. Thus we have reduced to the case $v = e_2 + \lambda e_{-2}$, and since w is perpendicular to v we may assume $w = e_1 + \mu e_{-1}$. But then $v \bullet w = e_2 + \mu^{2^n} e_1 + \lambda^{2^n} e_{-1} + (\lambda\mu)^{2^n} e_{-2}$ so $\lambda = 0$. Therefore there are precisely q^2 points with degree 2, and so $q^2 + 1$ points altogether.

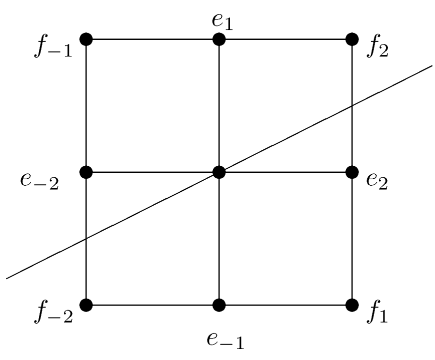
Moreover, the proof shows that every point of degree 2 can be mapped to $\langle e_2\rangle$ by an element of the Borel subgroup $B = \langle x, H\rangle$. Since B fixes the point $\langle e_{-2}\rangle$, and r interchanges e_{-2} with e_2, we have shown that G acts 2-transitively on the $q^2 + 1$ points. Now consider the stabiliser of the two points $\langle e_{-2}\rangle$ and $\langle e_2\rangle$. If these two points are fixed then so are the subspaces $\langle e_{-2}, e_2\rangle^\perp = \langle e_{-1}, e_1\rangle$ and $e_2 \bullet \langle e_{-1}, e_1\rangle = \langle e_2, e_1\rangle$, and their intersection $\langle e_1\rangle$. Similarly, $e_{-2} \bullet \langle e_{-1}, e_1\rangle = \langle e_{-2}, e_{-1}\rangle$ is fixed and therefore so is $\langle e_{-1}\rangle$. But as we observed above, every diagonal element of G lies in H, which has order $q-1$, so the 2-point stabiliser has order $q-1$, and the stabiliser of the pair of points is a dihedral group of order $2(q-1)$.

Therefore G has order $(q^2+1)q^2(q-1)$, and the point stabiliser has order $q^2(q-1)$. Since B has order at least $q^2(q-1)$, it follows that B is the full stabiliser in G of the point $\langle e_{-2}\rangle$. Moreover, G is generated by x, H and r, which concludes the proof that $G = \mathrm{Sz}(q)$.

In the case $n = 0$, that is $q = 2$, the group acts 2-transitively on the 5 points, and the point stabiliser is C_4. Thus $\mathrm{Sz}(2) \cong 5{:}4$.

Otherwise, the 2-point stabiliser is a cyclic group H of order $q-1$ which fixes no other points. By 2-transitivity, the 1-point stabiliser B is generated by conjugates of H, and therefore so is G. Since H is inverted by r, these generators are commutators, and so G is perfect. Also, B consists of lower triangular matrices, so is soluble. The permutation action of G on the q^2+1 points is 2-transitive, so primitive, and faithful. In other words, G is a finite perfect group acting faithfully and primitively on a set, such that the point stabiliser B has a soluble normal subgroup (namely, B itself) whose conjugates generate G. Therefore, by one of the standard variants of Iwasawa's Lemma, G is simple.

To see how my construction relates to the Lie theory, consider the following picture of the root system of type B_2.



The short roots are labelled by $e_{\pm1}$, $e_{\pm2}$, and the long roots by $f_{\pm1}$, $f_{\pm2}$, in such a way that reflection in the oblique line maps e_i to a scalar multiple of f_i. The non-zero terms in the symplectic form correspond to pairs of short roots which sum to zero. The non-zero terms of the bullet product correspond to pairs of short roots whose sum is a long root. More precisely, if $e_i + e_j = f_k$ in the root system, then $e_i \bullet e_j = e_k$ in V. The grading of the basis vectors corresponds to the ordering of the projections of the corresponding roots onto the oblique line.

I note also that the finiteness of the field F is not necessary for the definitions in this paper. We do however require that the field is a perfect field of characteristic 2, which means that the Frobenius endomorphism $\lambda \mapsto \lambda^2$ is an automorphism. (In fact, we can remove the requirement for the field to be perfect, at the expense of replacing the product $\pi\colon \ker\phi \to V$ by the corresponding coproduct $\pi'\colon V \to V \wedge V/(\ker\phi)^\perp$.) We also require that F has an automorphism σ which squares to the Frobenius automorphism, that is $\lambda^{\sigma^2} = \lambda^2$. The inverse of the Frobenius automorphism may be written $\lambda \mapsto \lambda^{1/2}$, and λ^σ may be written $\lambda^{\sqrt{2}}$. Any proofs which involve counting obviously do not go through, but we still obtain a parametrisation of the points of degree 2 by pairs of field elements, and a 2-transitive action on the points. Also, Iwasawa's Lemma extends to infinite groups in the case when the permutation action is 2-transitive, so the resulting groups are still simple.

Finally I remark that the approach taken in this paper can also be used to give elementary constructions of the two families of Ree groups (see [**5**] and [**6**]). In the case of the Ree groups of type G_2, reasonably elementary existence proofs already exist in the literature, but arguably this is not the case for the Ree groups of type F_4.

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