

Conjugacy class representatives in Fischer's Baby Monster

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Abstract

We determine a set of conjugacy class representatives for the elements in Fischer's Baby Monster simple group, up to algebraic conjugacy.

1 Introduction

Fischer's Baby Monster group is the second largest of the 26 sporadic simple groups, and has order greater than 4×10^{33} . Many of its basic properties were described by Fischer [3], and its character table was computed by Hunt [4] (see also [2]). It was first constructed by Leon and Sims [5], essentially as a permutation group on some 10^{10} points.

In [12] the author constructed the 4370-dimensional representation of the Baby Monster over $GF(2)$. In [8] the 4371-dimensional representation was constructed over $GF(3)$ and $GF(5)$. The method could in principle be applied also to construct the representation over any field of odd characteristic.

These matrix constructions now give almost enough invariants to distinguish all conjugacy classes of elements. In this paper we use this to produce a complete set of conjugacy class representatives for the Baby Monster.

Table 1: The elements a to z

a	b	$c = ab$	$d = cb$	$e = cd$	$f = ce$	$g = fc$	$h = gd$
2	3	55	55	40	20	12	18
$i = ch$	$j = id$	$k = jd$	$l = ck$	$m = lc$	$n = md$	$o = nd$	$p = eo$
31	23	17	46	16	46	24	34
$q = pe$	$r = qc$	$s = dr$	$t = sc$	$u = no$	$v = il$	$w = gh$	
36	60	26	47	47	34	47	
$x = (a(CG)^{24})^6 \quad y = ((CG)^6x)^{12} \quad z = ((CG)^4xcgx)^{15}$							
$\underline{\qquad\qquad\qquad 2 \qquad\qquad\qquad 2 \qquad\qquad\qquad 2 \qquad\qquad\qquad}$							

As usual, we note that it is sufficient to find representatives for the maximal cyclic subgroups, as conjugacy class representatives for the elements can then be found as suitable powers of the given generators of the maximal cyclic subgroups. It is easy to calculate from the class list and power maps that there are just 76 classes of maximal cyclic subgroups. The Atlas names for these are listed in Table 2. We also include in this table the main result of the paper, namely words for generators of such maximal cyclic subgroups, using the elements a to z defined in Table 1. This table also gives the orders of these elements. The derivation of the words in Table 2 is the main purpose of this paper.

2 Distinguishing conjugacy classes

Our main tool here is the character table (see [2]). All the classes of maximal cyclic subgroups of odd order are determined by the order. These are $25A$, $27A$, $31AB$, $39A$, $47AB$, and $55A$. Thus we concentrate on the classes of elements of even order from now on. We work as far as possible in the 2-modular representation, as that is much faster to work in than the others. As well as the order, we use the trace as a cheap but not very useful class invariant. We also calculate the dimension of the fixed space, which can be used later as a more discriminating invariant for even order elements. (This is of no use for odd-order elements, as it can already be calculated from the

Table 2: Maximal cyclic subgroups

12A	12H	12I	12L	12P	12S	12T	16E
$yz(CG)^{16}$	i^3vj	h^2n	$ehvk$	n^2snv	efj	cwh	api
16F	16G	16H	18A	18B	18D	18F	20B
dej	m	guk	mo	dtv	aoj	h	$clsh$
20C	20H	20I	20J	24A	24B	24C	24D
$cfvo$	fh^2	ij^2	fg^2	fmw	kuq	$cgon$	fu
24E	24F	24H	24I	24J	24K	24L	24M
efh	e^2fh	ps	fhj	fhg	cig	wgl	wgk
24N	25A	26B	27A	28A	28C	28D	28E
bv	cfg	cfh	hj	afr	h^2i	uw	gj
30A	30B	30C	30E	30GH	31AB	32AB	32CD
lum	agq	guq	hg^2	cjg	i	ci	cn
34A	34BC	36A	36B	36C	38A	39A	40A
v	fg	tw	hl	ehg	ij	cfj	nv^2
40B	40C	40D	40E	42A	42B	42C	44A
gj^2	op	e	cgh	amu	amo	gn	gi
46AB	47AB	48A	48B	52A	55A	56AB	60A
l	fi	cg	kn	e^2i	c	eih	djf
60B	60C	66A	70A				
nv	cg^2	gji	cih				

character table.) It is possible to calculate the full Jordan block structure, but this turns out to be of little use.

In addition to the classes of odd-order elements, the types $38A$, $44A$, $46AB$, $52A$, $56AB$, $66A$, and $70A$ are determined by their orders. Thus we only need to find elements of all these orders in order to find representatives for these classes and all their powers. Using the trace mod 2 as well, we can characterise the classes $60C$, $40E$, and $36C$. Using the trace mod 3, we can distinguish $48A$ and $48B$, $42B$, $32AB$ and $32CD$, $28A$, and $40B$. Using both traces, we can distinguish also $40A$, $30C$ and $30GH$.

Since by this stage we know involutions of each class, as suitable powers of elements already identified, we can calculate the codimensions of their fixed spaces modulo 2 as 1860, 2048, 2158 and 2168 for classes $2A$, $2B$, $2C$, $2D$, respectively. This enables us to distinguish certain classes by the class of involution they power up to. This deals with $26B$, $42C$, $34A$, $34BC$, $28E$, $20H$ and $18F$, and also $42A$, if we use the trace mod 3 as well.

For elements of order 60, we can distinguish between $60A$ and $60B$ by the trace mod 3 of the 5th power. For elements of order 40, we distinguish $40C$ and $40D$ from the rest by the trace mod 2 and mod 3, and then distinguish them from each other by the trace mod 3 of the 10th power. For elements of order 36, we distinguish $36A$ and $36B$ from $36C$ by the trace mod 2, and from each other by trace mod 5 of the 9th power.

For elements of order 30, we distinguish $30D$ and $30E$ from the rest by trace mod 3, and from each other by the trace mod 3 of the fifth power. Similarly, $30A$ and $30B$ can be distinguished from the rest by trace mod 2 and 3, and from each other by trace mod 5. For elements of order 28, the classes $28C$, $28D$ and $28E$ are distinguished by having trace 0 mod 3, and then the traces of their 7th powers are 1, 0, 2 respectively, mod 3.

For elements of order 24, the trace mod 2 and 5 separate them into seven sets $24A/I$, $24B/E/J$, $24C/G$, $24D$, $24F$, $24H/K/L$ and $24M/N$. Each of these sets can then be resolved by the trace mod 3 of the square. This deals with all elements of order 24. For elements of order 20, the trace mod 2 and mod 3 distinguishes $20B$, $20C$ and $20I$, and separates the rest into three classes, namely $20A/E/H$, $20D/F$ and $20G/J$, all of which can be distinguished by the trace of the fifth power mod 3.

For elements of order 18, the trace mod 2 and 3, and the 9th power, distinguish classes except for $18A$ and $18B$. The elements of order 16, by traces mod 3 alone are distinguished up to ambiguities $16A/D/F$, $16C/E$ and $16B/G/H$. The traces of the squares mod 3 resolve all of these except

the pair $16B/H$, which can be resolved by the trace mod 5, and the one remaining problem, that of distinguishing $16D/F$ (see below).

For elements of order 12, the trace mod 2, 3, and 5 distinguishes classes except for the following ambiguities: $12B/K/Q$, $12F/O$, $12H/P$, $12J/L$, $12M/R$, $12N/T$. The first is not required for our purposes, while the rest can be distinguished by the class of the 6th power (determined as above by the codimension of its fixed space).

3 Difficult cases

While most classes are fairly easily found by the above methods, there were a few problem cases. These were the classes $18A$ and $18B$, which cannot be distinguished by traces and power maps alone; $16F$, which similarly cannot be distinguished from $16D$; and $12A$, which is such a small conjugacy class that finding a $12A$ -element at random is impracticable.

3.1 The classes $18A$ and $18B$

We found two elements which are each in either $18A$ or $18B$, but one has fixed space of dimension 280 in the 4370-space over $GF(2)$, while the other has fixed space dimension 282. Thus one of them is in $18A$ and the other is in $18B$, but without extra information we cannot tell which is which. For example, we might look inside the involution centralizer: they each power up to a $2A$ -element, and inside the $2A$ -centralizer $2^2E_6(2):2$ one maps to a $9A$ -element and the other maps to a $9B$ -element. However, these elements are still not easy to distinguish in this subgroup.

An alternative approach is to look inside a subgroup Fi_{23} . We first find such a subgroup, with standard generators $(e^{20})c^{10}$ and $(g^4)^{d^9}$ given in term of the words in Table 1 in generators a and b of the Baby Monster. Words for representatives of all the conjugacy classes of elements in Fi_{23} are given in [11] (see also [14]), and we can test the elements of order 18. We found that an element of Fi_{23} -class $18A$ has fixed space dimension 282. This class corresponds to class $-9A$ in $2 \cdot Fi_{22}$, which fuses to class $-9A$ in $2^2E_6(2)$, and thence to $18A$ in B .

3.2 The class $16F$

To resolve the classes $16D$ and $16F$ we adopted a slightly different approach, which involved finding the centralizer order directly. Suppose we have an element x in one of these two classes, and work in $C(x^8) \cong 2^{1+22} \cdot Co_2$. Now x is conjugate to x^9 , so $C(x)$ has the same order as the centralizer of its image in $C(x^8)/(x^8)$. We calculate the latter as follows.

First calculate the involution centralizer (in the 2-modular representation) by one of the standard methods, and chop the representation to obtain an irreducible 22-dimensional constituent. Use the latter to find standard generators for the quotient Co_2 , and hence find words for a subgroup $U_6(2):2$ thereof.

Next switch to the 3-modular representation, and again chop the restriction to the involution centralizer. Take the 2300-dimensional constituent, and find the invariant 1-space of the group $2^{22} \cdot U_6(2):2$ which acts on it. A vector in this 1-space has 4600 images under $2^{22} \cdot Co_2$, which acts faithfully on this orbit. Therefore we can convert to a permutation representation of $2^{22} \cdot Co_2$ on 4600 points, and use GAP [10] to find the centralizer order of our element quickly.

We applied this to the element dej , and found that its centralizer had order 1024, and therefore it is a $16F$ -element.

3.3 The class $12A$

There is a problem with very rare classes, such as $12A$, where a purely random search would take a very long time. In this case, only about 1 in 4 million elements of the group are in the class $12A$, and it would take about 3 years of CPU time to make 4 million elements on a Pentium 4/1400. Indeed even with a rapid screening procedure to eliminate elements of order not 12 we estimated a CPU-time requirement of many months, and therefore decided on a different strategy. The result is a significantly longer word, which requires 30 multiplications to make, rather than the 5 multiplications we would expect from the random approach.

We first take an element which powers up to a $6C$ -element, and find the centralizer of the involution it powers to. For example, we can take the $48A$ -element cg , and find the centralizer of $(cg)^{24}$ to be generated by cg and $x = (a(cg)^{24})^6$. We now have a group $2^{1+22} \cdot Co_2$ and an element $(cg)^{16}$ mapping to Co_2 -class $3B$. We now search in the corresponding coset

of 2^{1+22} for elements of order 12, and use the trace mod 2, 3 and 5 to test for membership in class 12A.

4 Further remarks

In fact, we found our elements in various classes by employing other methods than those described above. The point is that if we can guess the class correctly, then it is easy to prove our guess is correct using the criteria in the previous section. We actually used various tables of partial information collected over the years (with a few mistakes in) to help us find elements in various classes, and then proved them as above.

Finally, having produced elements in each of the conjugacy classes, we can tabulate information which may not be easily obtainable by other means. In particular we have calculated the dimensions of the fixed spaces of all the elements of even order on the 4370-dimensional module over $GF(2)$. This is a useful conjugacy-class invariant which can often be used to identify the conjugacy class of a given element more quickly than the methods described above. We tabulate this information in Table 3.

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Table 3: Codimensions of fixed spaces

2A	1860	8G	3810	12Q	4002	20H	4144	30E	4214
2B	2048	8H	3780	12R	4002	20I	4138	30F	4224
2C	2158	8I	3786	12S	4004	20J	4150	30GH	4216
2D	2168	8J	3812	12T	4002	22A	4140	32AB	4222
4A	3114	8K	3818	14A	3996	22B	4158	32CD	4222
4B	3114	8L	3786	14B	4008	24A	4152	34A	4238
4C	3192	8M	3818	14C	4048	24B	4152	34BC	4220
4D	3192	8N	3818	14D	4034	24C	4152	36A	4226
4E	3256	10A	3860	14E	4052	24D	4152	36B	4238
4F	3202	10B	3896	16A	4072	24E	4164	36C	4248
4G	3204	10C	3918	16B	4072	24F	4170	38A	4236
4H	3266	10D	3908	16C	4074	24G	4164	40A	4242
4I	3264	10E	3920	16D	4074	24H	4182	40B	4242
4J	3266	10F	3932	16E	4074	24I	4176	40C	4242
6A	3486	12A	3936	16F	4074	24J	4178	40D	4250
6B	3510	12B	3942	16G	4094	24K	4174	40E	4258
6C	3566	12C	3936	16H	4094	24L	4186	42A	4242
6D	3534	12D	3936	18A	4088	24M	4176	42B	4242
6E	3606	12E	3958	18B	4090	24N	4186	42C	4258
6F	3604	12F	3996	18C	4110	26A	4196	44A	4254
6G	3596	12G	3962	18D	4124	26B	4176	46AB	4270
6H	3610	12H	3962	18E	4128	28A	4188	48A	4266
6I	3636	12I	3964	18F	4122	28B	4200	48B	4266
6J	3638	12J	3986	20A	4114	28C	4188	52A	4284
6K	3634	12K	3978	20B	4114	28D	4200	56AB	4284
8A	3774	12L	3966	20C	4114	28E	4210	60A	4280
8B	3738	12M	3978	20D	4114	30A	4190	60B	4286
8C	3738	12N	4000	20E	4128	30B	4190	60C	4296
8D	3778	12O	3982	20F	4132	30C	4212	66A	4286
8E	3738	12P	3988	20G	4148	30D	4206	70A	4292
8F	3780								

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