Another new approach to the small Ree groups

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Abstract. A new elementary construction of the small Ree groups is described.

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1. Introduction

The Ree groups of characteristic 3 were originally discovered in about 1960 by Rimhak Ree [3]. They were constructed as the fixed points of certain automorphisms of order 2 of the Dickson–Chevalley groups $G_2(3^{2n+1})$, acting on the Lie algebra of dimension 14. As this Lie algebra has a 7-dimensional ideal, it is possible to describe the Ree groups acting on a 7-dimensional vector space, or a 6-dimensional projective space. This is how Tits [5] described the group, in effect giving an independent existence proof, although the substantial calculations required were not published.

In [7] I presented a new construction of the Ree groups of characteristic 3, including complete calculations, using a moderately complicated object which I called the octopus algebra. Subsequently I have found a way of simplifying the construction substantially, which among other benefits removes a great deal of the calculations from the proofs. This new construction is presented here in the hope that in time it (or something very like it) will come to be regarded as the standard construction.

A similar construction of the Ree groups of characteristic 2 is described in [8], which provides the first genuinely elementary proof of existence of these groups, and is independent of Ree’s original proof [4]. The analogue for the Suzuki groups is even simpler [9], and provides a complete, elementary, and constructive proof of existence (and simplicity) of these groups in about three pages.

2. Definitions

First we set up the octonion algebra. Let n be a non-negative integer, let $F = \mathbb{F}_q$ be the field of order $q = 3^{2n+1}$, and let W be a 7-dimensional vector space over F with basis $\{e_i \mid -3 \leq i \leq 3\}$. Let b be the symmetric bilinear form defined on W by $b(e_i, e_{-i}) = 1$ and all other $b(e_i, e_j) = 0$. Let t be the alternating trilinear form on W defined by $t(e_i, e_j, e_k) = 0$ unless $i + j + k = 0$, and $t(e_i, e_j, e_k) = 1$ when (i, j, k) is a cyclic permutation of one of the triples $(0, 1, -1)$, $(0, 2, -2)$, $(0, -3, 3)$, $(1, 2, -3)$ or $(-1, -2, 3)$. [Recall that alternating means $t(u, v, w) = t(v, w, u)$ and $t(u, u, v) = 0$, so $t(v, u, w) = -t(u, v, w)$.] There is now a natural product $*$ on W defined by $t(u, v, w) = b(u * v, w)$. We call this the *octonion product* as it can be identified with the product on the pure imaginary part of a split octonion algebra. On the basis vectors, the non-zero products are given by $e_i * e_j = e_k, -e_k$ according as $t(e_i, e_j, e_{-k}) = 1, -1$. In particular the product is graded in the sense that $e_i * e_j \in \langle e_{i+j} \rangle$ (where this is interpreted as zero if $i + j$ is not in the range -3 to 3). To assist with calculations, I include the multiplication table for this product here. In this table, blank entries denote zero.

*	e_{-3}	e_{-2}	e_{-1}	e_0	e_1	e_2	e_3
e_{-3}				$-e_{-3}$	e_{-2}	$-e_{-1}$	e_0
e_{-2}			$-e_{-3}$	e_{-2}		$-e_0$	e_1
e_{-1}		e_{-3}		e_{-1}	$-e_0$		$-e_2$
e_0	e_{-3}	$-e_{-2}$	$-e_{-1}$		e_1	e_2	$-e_3$
e_1	$-e_{-2}$		e_0	$-e_1$		e_3	
e_2	e_{-1}	e_0		$-e_2$	$-e_3$		
e_3	$-e_0$	$-e_1$	e_2	e_3			

The new ingredient of the construction is an alternating *bullet product* $u \bullet v$ defined whenever u and v are isotropic vectors with $u * v = 0$, satisfying the twisted linearity law

$$(u + \lambda v) \bullet w = u \bullet w + \lambda^{3^n} (v \bullet w).$$

In order to see that $\bullet$ is well-defined, it is easiest to reinterpret the octonion product as a linear map κ from $W \wedge W$ to W , and then reinterpret $\bullet$ as a map π from the kernel of κ to W . The twisted linearity conditions imply that it is necessary and sufficient to define π on a basis of $\ker \kappa$, which has dimension 14. The equations below do exactly this. The bullet product is invariant under the symmetry $r : e_0 \mapsto -e_0, e_i \mapsto e_{-i}$ for $i \neq 0$, and is defined by the rules

$$\begin{aligned} e_3 \bullet e_2 &= e_3 \\ e_1 \bullet e_3 &= e_2 \\ e_2 \bullet e_{-1} &= e_1 \\ e_{-1} \bullet e_1 + e_2 \bullet e_{-2} &= e_0 \end{aligned}$$

[The last line of the display is shorthand for $(e_{-1} + e_{-2}) \bullet (e_1 - e_2) = e_1 + e_0 + e_{-1}$, cancelling out the cross terms, since $e_{-1} \bullet e_1$ and $e_{-2} \bullet e_2$ are not themselves

defined.] All ‘other’ products are zero, in the sense that

$$\begin{aligned} e_0 \bullet e_1 + e_3 \bullet e_{-2} &= 0 \\ e_0 \bullet e_2 + e_{-1} \bullet e_3 &= 0 \\ e_0 \bullet e_3 + e_1 \bullet e_2 &= 0 \\ e_{-3} \bullet e_3 + e_2 \bullet e_{-2} + e_1 \bullet e_{-1} &= 0 \end{aligned}$$

These equations in (strictly) undefined products are equivalent to the following equations in well-defined products, expanded and with known terms cancelled out:

$$\begin{aligned} (e_1 + e_0 + e_{-1}) \bullet (e_{-2} - e_{-3}) &= -e_{-1} - e_{-2} \\ (e_1 + e_0 + e_{-1}) \bullet (e_3 + e_2) &= e_2 - e_1 \\ (e_2 + e_0 + e_{-2}) \bullet (e_3 - e_1) &= -e_3 - e_{-1} \\ (e_2 + e_0 + e_{-2}) \bullet (e_{-1} + e_{-3}) &= e_1 - e_{-3} \\ (e_3 + e_0 + e_{-3}) \bullet (e_2 - e_{-1}) &= e_3 + e_{-2} \\ (e_3 + e_0 + e_{-3}) \bullet (e_1 + e_{-2}) &= -e_2 + e_{-3} \\ (e_3 + e_{-2} + e_{-1}) \bullet (e_2 + e_1 + e_{-3}) &= e_3 - e_2 - e_1 + e_{-1} + e_{-2} - e_{-3} \end{aligned}$$

In order to assist with calculations, I include a multiplication table for this product here. In this table, blank entries may be taken to be zero, since in any permissible calculation they will always cancel out. The terms $e_i \bullet e_{-i}$ may be taken to be $\pm e_0$ in various ways. One choice is given in the table.

$\bullet$	e_{-3}	e_{-2}	e_{-1}	e_0	e_1	e_2	e_3
e_{-3}		e_{-3}	$-e_{-2}$				$-e_0$
e_{-2}	$-e_{-3}$				e_{-1}	$-e_0$	
e_{-1}	e_{-2}					$-e_1$	
e_0							
e_1		$-e_{-1}$					e_2
e_2		e_0	e_1				$-e_3$
e_3	e_0				$-e_2$	e_3	

Indeed, the non-zero values of this product are best summarised in the following table.

$\bullet$	e_{-3}	e_{-1}	e_2
e_{-2}	$-e_{-3}$	e_{-1}	$-e_0$
e_{-1}	e_{-2}	0	$-e_1$
e_3	e_0	$-e_2$	e_3

Let $\mathbb{W}$ denote the algebraic structure consisting of the vector space W endowed with the bilinear form b , the trilinear form t , and the partial product $\bullet$. Let $R(q)$ be the automorphism group of $\mathbb{W}$, that is, the group of linear maps on W which preserve b , t and $\bullet$. The geometry $\mathcal{G}$ on which it acts consists of the *points* $\langle v \rangle$ defined by the property that $v = v \bullet w$ for some w . (If there is danger of confusion, I will call these points *bullet points*, since they are defined by the bullet product.) The main result of this paper is that $R(q)$ is isomorphic to the

Ree group as usually defined. More specifically, our results are summarised in the following theorem.

- Theorem 1.**
1. *The geometry $\mathcal{G}$ consists of $q^3 + 1$ points.*
 2. *The group $R(q)$ acts 2-transitively on the points of $\mathcal{G}$.*
 3. *$R(q)$ has order $q^3(q^3 + 1)(q - 1)$.*
 4. *$R(q)$ permutes the $q^6 + q^3$ vectors of norm 1 in W transitively.*
 5. *The centraliser of an involution in $R(q)$ is $2 \times \text{PSL}_2(q)$.*
 6. *The Sylow 2-normaliser in $R(q)$ is $2^3:7:3$.*
 7. *The normaliser of a 2^2 in $R(q)$ is $(2^2 \times D_{(q+1)/2}):3$.*
 8. *If $q > 3$ then $R(q)$ is simple.*
 9. *$R(3) \cong \text{PSL}_2(8):3$.*
 10. *$R(q)$ is isomorphic to the Ree group as usually defined.*

The rest of this paper is devoted to proving these facts.

3. Proofs

We have already observed that the forms b and t (and therefore the octonion product $*$) respect the grading of the coordinates, in the sense that $b(e_i, e_j) = 0$ unless $i + j = 0$, and $t(e_i, e_j, e_k) = 0$ unless $i + j + k = 0$. The product $\bullet$ also respects the grading in the sense that (whenever $e_i \bullet e_j$ is defined and non-zero) $e_i \bullet e_j \in \langle e_{g(i+j)} \rangle$, where g is the monotonic function taking $-5, -4, -1, 0, 1, 4, 5$ to $-3, -2, -1, 0, 1, 2, 3$ respectively. This means that it makes sense to talk about the *leading term* of a vector $v = \sum_{i=-3}^3 \lambda_i e_i$, that is the term e_i for the maximum value of i for which $\lambda_i \neq 0$. If $u \bullet v = w$ then without loss of generality the leading terms of u and v are different, and the leading term of w is equal to the product of the leading terms of u and v . In particular, if $v \bullet w = v$, then the leading term of v occurs on both sides of one of the defining equations $e_i \bullet e_j = \pm e_k$, so by inspection is either e_{-3} or e_3 .

Next we investigate the stabiliser of $\langle e_0 \rangle$.

Lemma 1. *The stabiliser in $R(q)$ of $\langle e_0 \rangle$ is a dihedral group of order $2(q - 1)$, generated by the diagonal elements*

$$d(\lambda) = \text{diag}(\lambda^{-3^{n+1}-2}, \lambda^{-3^{n+1}-1}, \lambda^{-1}, 1, \lambda, \lambda^{3^{n+1}+1}, \lambda^{3^{n+1}+2})$$

and the involution $r : e_0 \mapsto -e_0, e_i \mapsto e_{-i}$ for $i \neq 0$.

Proof. Observe that the map $v \mapsto v * e_0$ is diagonalisable, and has three eigenvalues 0, -1 , and 1 , with corresponding eigenspaces $V_0 = \langle e_0 \rangle$, $V_{-1} = \langle e_{-3}, e_1, e_2 \rangle$ and $V_1 = \langle e_3, e_{-1}, e_{-2} \rangle$. Now V_{-1} contains a unique (bullet) point, namely $\langle e_{-3} \rangle$, since any bullet point in V_{-1} has leading term e_{-3} . Therefore by symmetry V_1 contains a unique point, namely $\langle e_3 \rangle$. In particular, the stabiliser of the vector e_0 fixes the 1-spaces $\langle e_{-3} \rangle$ and $\langle e_3 \rangle$, so also fixes the spaces $V_1 \cap (e_{-3} \bullet W) = \langle e_{-2} \rangle$ and similarly $\langle e_2 \rangle$. Therefore it also fixes $V_1 \cap (e_{-2} \bullet W) = \langle e_{-1} \rangle$ and similarly $\langle e_1 \rangle$, so acts diagonally on the given basis.

Now any diagonal element $e_i \mapsto \lambda_i e_i$ which preserves b has $\lambda_{-i} = \lambda_i^{-1}$. If it also preserves t , then $\lambda_i \lambda_0 \lambda_{-i} = 1$ and therefore $\lambda_0 = 1$; and $\lambda_1 \lambda_2 = \lambda_3$. To preserve $\bullet$ it must also satisfy $(\lambda_2 \lambda_{-1})^{3^n} = \lambda_1$, that is $\lambda_2 = \lambda_1^{3^{n+1}+1}$. Conversely, it is easy to check that if these equations hold then the given diagonal element preserves b , t and $\bullet$. These elements form a cyclic group of order $q-1$, which is therefore the stabiliser of e_0 . Consequently, the stabiliser of $\langle e_0 \rangle$ is a dihedral group of order $2(q-1)$. $\square$

Next we study the stabiliser, B say, of $\langle e_{-3} \rangle$. Notice first that if v is any isotropic vector in V_1 , then $v \bullet W = v \bullet V_{-1} = v \bullet (V_{-1} \cap v^\perp)$ and is therefore spanned by two vectors $v \bullet w_1$ and $v \bullet w_2$, where w_1, w_2 span the 2-space $V_{-1} \cap v^\perp$. Similarly for $v \in V_{-1}$, and $\langle w_1, w_2 \rangle = V_1 \cap v^\perp$. Now we see that the group B must fix $e_{-3} \bullet W = \langle e_{-3} \bullet e_{-2}, e_{-3} \bullet e_{-1} \rangle = \langle e_{-3}, e_{-2} \rangle$, and therefore also fixes $\langle e_{-3}, e_{-2} \rangle \bullet W = \langle e_{-3} \bullet e_{-2}, e_{-3} \bullet e_{-1}, e_{-2} \bullet e_{-1} \rangle = \langle e_{-3}, e_{-2}, e_{-1} \rangle$. It also fixes their orthogonal complements, and therefore acts as lower triangular matrices with respect to the given ordered basis. In particular, $\langle e_0 \rangle$ maps to one of at most q^3 possible 1-spaces, so by Lemma 1 the stabiliser of $\langle e_{-3} \rangle$ has order at most $q^3(q-1)$. We shall show that the stabiliser has exactly this order by exhibiting symmetries which map e_0 to $e_0 + e_{-1}$, or $e_0 + e_{-2}$, or $e_0 + e_{-3}$. Indeed the method used in Lemma 1 to calculate the stabiliser of e_0 enables us to complete any map of e_0 to $e_0 +$ (lower terms) to a unique map on the whole space.

Lemma 2. *The stabiliser of e_{-3} in $R(q)$ is transitive on the q^3 vectors of shape $e_0 + \lambda_{-1}e_{-1} + \lambda_{-2}e_{-2} + \lambda_{-3}e_{-3}$.*

Proof. Consider first the most difficult case, $e_0 \mapsto e_0 + e_{-1}$, and write e'_i for the image of e_i under this map. We assume that e_{-3} is fixed, and then calculate the images of the remaining basis vectors $e_{-2}, e_{-1}, e_1, e_2, e_3$ in that order. The eigenspaces of octonion multiplication by $e'_0 = e_0 + e_{-1}$ are $V'_0 = \langle e_0 + e_{-1} \rangle$, $V'_1 = \langle e_3 - e_2, e_{-1}, e_{-2} + e_{-3} \rangle$ and $V'_{-1} = \langle e_{-3}, e_2, e_1 - e_0 + e_{-1} \rangle$. Since e_{-3} is fixed, e'_{-2} lies in $(e_{-3} \bullet W) \cap V'_1 = \langle e_{-2} + e_{-3} \rangle$. Also, $e'_{-3} \bullet e'_{-2} = e'_{-3}$, so $e'_{-2} = e_{-2} + e_{-3}$. Then

$$\begin{aligned} (e_{-2} + e_{-3}) \bullet W &= (e_{-2} + e_{-3}) \bullet \langle e_{-3}, e_1 - e_0 + e_{-1} \rangle \\ &= \langle e_{-3}, e_{-1} - e_{-2} \rangle \end{aligned}$$

which intersects V'_1 in $\langle e_{-1} - e_{-2} - e_{-3} \rangle$, and since $e'_{-1} \bullet e'_{-3} = e'_{-2}$ we have $e'_{-1} = e_{-1} - e_{-2} - e_{-3}$. Next

$$\begin{aligned} (e_{-1} - e_{-2} - e_{-3}) \bullet W &= (e_{-1} - e_{-2} - e_{-3}) \bullet \langle e_{-3}, e_2 + e_1 - e_0 + e_{-1} \rangle \\ &= \langle e_{-2} + e_{-3}, -e_1 - e_{-1} + e_0 + e_{-2} \rangle \end{aligned}$$

which intersects V'_{-1} in $\langle e_1 - e_0 + e_{-1} + e_{-3} \rangle$, and the correct scalar multiple is determined by the requirement that $e'_{-2} \bullet e'_1 = e'_{-1}$. Then

$$\begin{aligned} (e_1 - e_0 + e_{-1} + e_{-3}) \bullet W &= (e_1 - e_0 + e_{-1} + e_{-3}) \bullet \langle e_{-2} + e_{-3}, e_3 - e_2 - e_{-1} \rangle \\ &= \langle -e_{-1} + e_{-2} + e_{-3}, e_2 + e_1 - e_0 + e_{-2} \rangle \end{aligned}$$

which intersects V'_{-1} in $\langle e_2 + e_1 - e_0 + e_{-1} - e_{-3} \rangle$, and the correct scalar multiple is given by $e'_2 \bullet e'_{-1} = e'_1$. This gives us all but the last row of the matrix, which is then determined by orthogonality, as follows:

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ -1 & -1 & 1 & & & & \\ 0 & 0 & 1 & 1 & & & \\ 1 & 0 & 1 & -1 & 1 & & \\ -1 & 0 & 1 & -1 & 1 & 1 & \\ -1 & -1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Conversely, we need to check that this matrix preserves the algebraic structure. To check that it preserves b , we easily check that V'_0 is orthogonal to V'_1 and V'_{-1} , and that V'_1 and V'_{-1} are totally isotropic. Finally, the bases $\{e'_{-2}, e'_{-1}, e'_3\}$ for V'_1 and $\{e'_2, e'_1, e'_{-3}\}$ for V'_{-1} are easily checked to be dual to each other.

To check that the trilinear form t (or, equivalently, the octonion product $*$) is preserved, notice that by construction the products with e'_0 are correct. This deals with all terms of the trilinear form involving e'_0 . Every other term involves either two vectors from V'_1 or two vectors from V'_{-1} , so it suffices to consider products of such pairs of vectors. Now we calculate

$$\begin{aligned} e'_{-2} * e'_{-1} &= (e_{-2} + e_{-3}) * (e_{-1} - e_{-2} - e_{-3}) \\ &= (e_{-2} + e_{-3}) * e_{-1} = -e_{-3} = -e'_{-3} \\ e'_{-2} * e'_3 &= (e_{-2} + e_{-3}) * (e_3 - e_2 - e_{-1} - e_{-2} - e_{-3}) \\ &= e_1 - e_0 + e_{-1} + e_{-3} = e'_1 \\ e'_{-1} * e'_3 &= (e_{-1} - e_{-2} - e_{-3}) * (e_3 - e_2 - e_{-1} - e_{-2} - e_{-3}) \\ &= -e_2 - e_1 + e_0 - e_{-1} + e_{-3} = -e'_2 \\ e'_{-3} * e'_1 &= e_{-3} * (e_1 - e_0 + e_{-1} + e_{-3}) = e_{-2} + e_{-3} = e'_{-2} \\ e'_{-3} * e'_2 &= e_{-3} * (e_2 + e_1 - e_0 + e_{-1} - e_{-3}) = -e_{-1} + e_{-2} - e_{-3} = -e'_{-1} \\ e'_1 * e'_2 &= (e_1 - e_0 + e_{-1} + e_{-3}) * (e_2 + e_1 - e_0 + e_{-1} - e_{-3}) \\ &= e_3 + e_1 + e_0 + e_{-2} - e_2 - e_1 + e_{-1} + e_{-3} - e_0 - e_{-1} \\ &\quad - e_{-1} + e_{-2} + e_{-3} = e'_3 \end{aligned}$$

which by the cyclic symmetry of t is enough.

Checking the bullet product is a little more time-consuming, as we have to check the 14 defining equations individually. The first six are as follows.

$$\begin{aligned} e'_{-3} \bullet e'_{-2} &= e_{-3} \bullet (e_{-2} + e_{-3}) = e_{-3} = e'_{-3} \\ e'_{-1} \bullet e'_{-3} &= (e_{-1} - e_{-2} - e_{-3}) \bullet e_{-3} = e_{-2} + e_{-3} = e'_{-2} \\ e'_{-2} \bullet e'_1 &= (e_{-2} + e_{-3}) \bullet (e_1 - e_0 + e_{-1} + e_{-3}) = e_{-1} - e_{-3} - e_{-2} = e'_{-1} \\ e'_2 \bullet e'_{-1} &= (e_2 + e_1 - e_0 + e_{-1} - e_{-3}) \bullet (e_{-1} - e_{-2} - e_{-3}) \\ &= e_1 - e_0 + e_{-1} + e_{-2} + e_{-3} - e_{-2} = e'_1 \\ e'_1 \bullet e'_3 &= (e_1 - e_0 + e_{-1} + e_{-3}) \bullet (e_3 - e_2 - e_{-1} - e_{-2} - e_{-3}) \\ &= -e_0 + e_2 + e_{-2} + e_{-1} - e_{-3} + e_1 - e_{-2} = e'_2 \\ e'_3 \bullet e'_2 &= (e_3 - e_2 - e_{-1} - e_{-2} - e_{-3}) \bullet (e_2 + e_1 - e_0 + e_{-1} - e_{-3}) \\ &= e_3 - e_2 + e_1 - e_{-1} + e_{-2} - e_{-3} - e_1 + e_{-2} = e'_3 \end{aligned}$$

The other eight calculations are similar and left as an exercise for the reader.

Similar calculations produce symmetries which extend the maps $e_0 \mapsto e_0 + e_{-2}$ and $e_0 \mapsto e_0 + e_{-3}$. The resulting matrices are

$$\begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ -1 & 0 & 1 & & & & \\ 0 & 1 & 0 & 1 & & & \\ 1 & 0 & 0 & 0 & 1 & & \\ 0 & 1 & 0 & -1 & 0 & 1 & \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ 1 & 0 & 0 & 1 & & & \\ 0 & -1 & 0 & 0 & 1 & & \\ 1 & 0 & 1 & 0 & 0 & 1 & \\ 1 & -1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Conjugating these and their inverses by suitable diagonal elements, and multiplying them together, gives the general case. $\square$

It follows immediately that the stabiliser of e_{-3} has order q^3 , and the stabiliser of the bullet point $\langle e_{-3} \rangle$ has order $q^3(q-1)$. We next show that there are exactly $q^3 + 1$ bullet points.

Lemma 3. *The number of (bullet) points is $q^3 + 1$, and $R(q)$ permutes them transitively.*

Proof. Observe from the multiplication table that if v and w are isotropic vectors with octonion product $v * w = 0$, then v and w are mutually orthogonal. Suppose now that $\langle v \rangle$ is a point with leading term e_3 , so that $v = v \bullet w$ and w has leading term e_2 . Then from the proof of the previous lemma we know that for every $\lambda \in \mathbb{F}_q$ there is an element of $R(q)$ mapping $e_3 \mapsto e_3 + \lambda e_2$ so we may assume that v has no term in e_2 . Similarly, applying a suitable symmetry $e_3 \mapsto e_3 + \lambda e_1$ we may assume v has no term in e_1 , and then we may apply a symmetry $e_3 \mapsto e_3 + \lambda e_0$ to remove the term in e_0 . Therefore $v = e_3 + \lambda_{-1}e_{-1} + \lambda_{-2}e_{-2} + \lambda_{-3}e_{-3}$. But v is isotropic, so $\lambda_{-3} = 0$. Now applying the symmetry $r : e_i \mapsto e_{-i}, e_0 \mapsto -e_0$ we obtain a point whose leading term is not e_3 . This point must be $\langle e_{-3} \rangle$, so $v = e_3$. Therefore every point is either $\langle e_{-3} \rangle$, or one of the q^3 images of $\langle e_3 \rangle$ under the action of the stabiliser of $\langle e_{-3} \rangle$. $\square$

This proves the first part of the main theorem. Since the stabiliser of the two points $\langle e_{-3} \rangle$ and $\langle e_3 \rangle$ has order $q-1$, the point stabiliser (of order $q^3(q-1)$) must be transitive on the remaining q^3 points. Hence $R(q)$ acts 2-transitively on $\mathcal{G}$, proving the second part of the theorem. It follows immediately that the order of $R(q)$ is $q^3(q^3+1)(q-1)$, which is the third part of the theorem. Since the stabiliser of the vector e_0 of norm 1 has order $q-1$, it follows that $R(q)$ is transitive on the $q^6 + q^3$ vectors of norm 1 in W , which is the fourth part of the theorem.

Since $q \equiv 3 \pmod{8}$, we have $q^3 + 1 \equiv 4 \pmod{8}$ and $q - 1 \equiv 2 \pmod{4}$ and therefore the Sylow 2-subgroup of $R(q)$ has order 8. Next consider the centraliser of the involution $d(-1) = \text{diag}(-1, 1, -1, 1, -1, 1, -1)$. This involution fixes precisely $q + 1$ points, namely $\langle e_{-3} \rangle$ and the q images of $\langle e_3 \rangle$ under the maps defined by

$e_{-3} \mapsto e_{-3}, e_0 \mapsto e_0 + \lambda e_{-2}$. Moreover its centraliser in the point stabiliser is a group of order $q(q-1)$ and shape $2 \times 3^{2n+1}:C_{q-1}$. Therefore its centraliser in $R(q)$ has order $q(q-1)(q+1)$. Now on the fixed 3-space $\langle e_{-2}, e_0, e_2 \rangle$ there are elements acting as

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \lambda^{-2} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where $\lambda \in \mathbb{F}_q$. These matrices generate $\Omega_3(q) \cong \text{PSL}_2(q)$. Since $R(q)$ contains an elementary abelian group of order 4, while $\text{SL}_2(q)$ does not, it follows that the centraliser of the given involution is $2 \times \text{PSL}_2(q)$. This is the fifth part of the theorem.

In particular, the Sylow 2-subgroup is elementary abelian of order 8, and its normaliser contains $2 \times A_4$. The latter group contains three classes of involutions, and it is easy to check that each involution fixes at least one point. Specifically, the element r fixes the point $e_3 - e_1 - e_0 + e_{-1} - e_{-3}$, while $d(-1)r$ fixes the point $e_3 - e_2 + e_1 + e_0 + e_{-1} + e_{-2} + e_{-3}$. But all involutions which fix a point are conjugate. Therefore all involutions in $R(q)$ are conjugate, and the Sylow 2-normaliser is transitive on its non-trivial elements, so has order 168 and shape $2^3:7:3$. This is the sixth part of the theorem. The seventh part follows immediately from the fact that the centraliser of a 2^2 is $2 \times D_{q+1}$, inside $2 \times \text{PSL}_2(q)$.

Simplicity of $R(q)$ for $q > 3$ follows from Iwasawa's Lemma as in [7]. For completeness we give the proof again here. We have shown that $R(q)$ acts faithfully and 2-transitively (therefore primitively) on a set of $q^3 + 1$ points. The point stabiliser is a soluble group of order $q^3(q-1)$. Provided $q > 3$, the point stabiliser is generated by the elements of order $q-1$, and therefore so is $R(q)$. Each such element is the product of an involution and an element of odd order $(q-1)/2$. The involutions are commutators already in $2^3:7:3$, and the elements of order $(q-1)/2$ are commutators inside $D_{2(q-1)}$. Hence $R(q)$ is perfect (when $q > 3$), and by Iwasawa's Lemma is simple.

In the case $q = 3$, the Sylow 2-normaliser $2^3:7:3$ has index 9 in $R(q)$, so $R(q)$ embeds in S_9 as a 3-transitive group on 9 points. Therefore $R(3) \cong \text{PSL}_2(8):3$, which is the ninth part of the theorem. To see that our groups $R(q)$ are isomorphic to the Ree groups as traditionally defined, we could invoke the Classification Theorem for Finite Simple Groups, or more specifically the classification of simple groups with involution centraliser $2 \times \text{PSL}_2(3^{2n+1})$ (see [1]). Perhaps more satisfactorily, one can calculate explicit formulae for the $q^3 + 1$ points and observe that they are essentially the same as those calculated by Tits [5] (see also [6]). Nevertheless, this is not a particularly easy calculation to do, and an alternative is to observe that our matrix generators are the same (up to some sign-changes on the basis vectors) as those calculated from the Lie theory by several people, including Howlett, Rylands and Taylor [2].

4. Further remarks

The definitions of both the octonion product $*$ and the partial product $\bullet$ can be motivated by consideration of the root system of type G_2 .

In this picture, if e_i and e_j are short roots whose sum is a short root e_k , and e_j is reached from e_i by turning clockwise through $1/3$ of a rotation, then $e_i * e_j = e_k$. Similarly, if e_i and e_j are short roots whose sum is a long root f_k , and e_j is reached from e_i by turning clockwise through $1/6$ of a rotation, then $e_i \bullet e_j = e_k$. The map between the short and long roots is given by reflection in the line at $\pi/4$ to the vertical, followed by the appropriate scaling. The grading of the coordinates is given by the ordering of their projections onto this line.

If we relabel the roots by complex numbers, so that e_1 becomes e_ω and e_{-3} becomes $e_{\bar{\omega}}$ while e_2 becomes e_1 , then we can write the above rules as $e_{\bar{\omega}z} * e_{\omega z} = e_{-z}$ and $e_{\omega z} \bullet e_{-\bar{\omega}z} = e_{\bar{z}}$.

The construction works equally well for infinite fields, provided there is an automorphism σ which squares to the inverse of the Frobenius automorphism. In particular, this requires the field to be perfect. By recasting the partial product as a coproduct (only defined modulo the octonion coproduct) we can remove the requirement for the field to be perfect, provided only that the Frobenius endomorphism has a square root.

It is easy to see that $R(q)$ has exactly three orbits on the $q^6 - 1$ isotropic vectors in W : the orbit containing e_{-3} has length $(q - 1)(q^3 + 1)$, while the orbit containing e_{-2} has length $(q - 1)q(q^3 + 1)$ and the orbit containing e_{-1} has length $(q - 1)q^2(q^3 + 1)$. The $q^6 - q^3$ vectors of norm -1 in fact fall into three orbits also, and the corresponding 1-space stabilisers are maximal (if $q > 3$) subgroups of $R(q)$ of shapes $(2^2 \times D_{(q+1)/2}):3$, $(q + \sqrt{3q} + 1):6$ and $(q - \sqrt{3q} + 1):6$. The first of these

is represented by the 1-space $\langle e_{-2} + e_2 \rangle$, whose stabiliser is the normaliser of the group $\langle r, d(-1) \rangle$.

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