Exceptional simplicity

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INTRODUCTION
What do mathematicians do?

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If you ask mathematicians what they do, you always get the same answer. They think. They think about difficult and unusual problems. They do not think about ordinary problems: they just write down the answers.

—M. Egrafov
On mathematicians

One of the endearing things about mathematicians is the extent to which they will go to avoid doing any real work.

—Matthew Pordage
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You know, we all became mathematicians for the same reason: we were lazy

—Max Rosenlicht
SYMMETRY
AND
GROUPS
What is group theory?

- The (abstract) study of symmetry
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...what a wealth, what a grandeur of thought may spring from what slight beginnings.

—H. F. Baker
Bilateral (or mirror) symmetry

Photo courtesy of Lloyd Carr
Bilateral (or mirror) symmetry
The language of group theory

The group of symmetries has order 2, meaning there are two elements of symmetry (i.e. “leave it alone”, and “reflect in the vertical axis”).
The equilateral triangle

has mirror symmetry, but now in three different axes.

There are six symmetries in all: three reflections, two rotations, and the “do nothing” symmetry.
The kaleidoscope is made with two mirrors set at $60^\circ$ to each other, so that by repeated reflections you get round the full circle.
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On special cases

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The art of doing mathematics consists in finding that special case which contains all the germs of generality.

—David Hilbert
Important properties of symmetry

- you can follow one symmetry \( r \) by another \( s \), to form the composite symmetry, called \( r + s \) (read \( r \) and then \( s \))
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- “Newton’s law”: every symmetry $r$ has an equal and opposite symmetry $s$, which “undoes” it. Thus $r + s = s + r = 0$.
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- “Newton’s law”: every symmetry $r$ has an equal and opposite symmetry $s$, which “undoes” it. Thus $r + s = s + r = 0$.
- “$r$ then $s$ then $t$” is the same whichever way you read it: $(r + s) + t = r + (s + t)$
A set of elements with these properties is called a group. Fact: Every group is the group of symmetries of some object.
Subgroups

One rotation followed by another makes a third rotation (or brings it back to where it started). We express this by saying

*the rotations form a subgroup.*

In total there are

- 3 rotations (including “leave it alone”), forming a subgroup of order 3
- 3 reflections, *not* forming a subgroup
REFLECTION GROUPS
Kaleidoscope groups

- Two mirrors, at some angle $A$. 

Examples:

- $n = 2$ gives $A = 90^\circ$
- $n = 3$ gives $A = 60^\circ$
Kaleidoscope groups

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- Two reflections combine to give a rotation through an angle $2A$. 

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Kaleidoscope groups, 2

- three vertical mirrors, at angles $A, B, C$
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- three vertical mirrors, at angles $A$, $B$, $C$
- angles are $180/a$ and $180/b$ and $180/c$ for some whole numbers $a$, $b$ and $c$. 
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- angles are $180/a$ and $180/b$ and $180/c$ for some whole numbers $a$, $b$ and $c$.
- But the angles in a triangle add up to 180 so $1/a + 1/b + 1/c = 1$
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- angles are $180/a$ and $180/b$ and $180/c$ for some whole numbers $a$, $b$ and $c$.
- But the angles in a triangle add up to 180 so $1/a + 1/b + 1/c = 1$
- the only possibilities are $(3, 3, 3)$ (an equilateral triangle), $(2, 3, 6)$ (half an equilateral triangle) and $(2, 4, 4)$ (a right-angled isosceles triangle).
The $(3,3,3)$ reflection group

Take the *fundamental region* surrounded by the three mirrors, and reflect it in all the mirrors. And then reflect the reflections in the other mirrors, and so on.
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3-dimensional kaleidoscopes

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3-dimensional kaleidoscopes

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- each pair of mirrors has to be a kaleidoscope
- the only combinations which work are \((2, 3, 3), (2, 3, 4)\) and \((2, 3, 5)\).
- These give you 3-dimensional kaleidoscopes with symmetries of
  - a tetrahedron,
  - a cube or octahedron,
  - a dodecahedron or icosahedron.
The Platonic solids

The last of these is also the symmetry group of a football. There are exactly 120 copies of the fundamental region, of which 60 are ‘right-handed’ and 60 are ‘left-handed’. This means the reflection group has order 120, and the rotation subgroup has order 60.
An 8 dimensional kaleidoscope

We put together 8 mirrors, at angles of either 60° or 90° to each other, according the rule given by this diagram:

- Each blob represents a mirror
- if there is a line between two mirrors, they are at 60°, and if not, they are at 90°.
- This 8-dimensional kaleidoscope is called $E_8$
A 2-dimensional picture of $E_8$

Picture courtesy of John Stembridge
A 3-dimensional picture of $E_8$
Properties of $E_8$

- The total number of mirror symmetries is 120
- The total number of symmetries of this 8-dimensional crystal is 696729600
- This is the same as the number of reflected copies of the fundamental region.
SIMPLE GROUPS
Quotient groups

In mathematics you don’t understand things, you just get used to them.

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The Star of David

- The symmetry group of the star has order 12

- It has a quotient group of order 2 swapping the colours: red and blue

- The colour-fixing subgroup has order 6

- This gives a 'factorisation' of the group

- And now we can do it again: the symmetry group of the triangle can be factored into
  - a subgroup of order 3 (the rotation subgroup) and
  - a quotient of order 2 (the 'flipping' group).
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- They are not necessarily easy.
- Reflection groups can never be simple (unless there’s only one reflection),
- But the rotation subgroup often is simple.
- For example, the rotation group of a regular $n$-gon is simple whenever $n$ is a prime number.
- The rotation group of the football is also simple. We call it $A_5$. 
Another quotient group

- Take the hexagonal tesselation of the plane, and imagine rolling it up into a cylinder.
- Now the horizontal translations have finite order.
- If the cylinder is made of rubber, we can roll it up again into a doughnut shape. Then the whole symmetry group is now finite.
- Here is an example, where the full translation group now has order 7, and the rotation group has order 6. There are no longer any reflection symmetries.
Hexagonal tessellation
Hexagonal tessellation
Hexagonal tessellation rolled up

Pictures courtesy of Richard Barraclough
The Classification of the Finite Simple Groups

Theorem
Every (finite) simple group is one of:
- a group of prime order
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- $A_5, A_6, A_7, \ldots$, i.e. $A_n$ for $n$ at least 5
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- in various other infinite families of groups
  - six families of classical groups
  - ten families of exceptional groups
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- 26 sporadic simple groups
Sporadic groups

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- The largest sporadic simple group is the Monster, discovered in 1973, which has order
  \[ 8080174247945128758864599049617107570057543680000000000 \]
Platonism

- Plato said: mathematics is discovered, not invented.
- apparently he was led to this conclusion by the existence of the dodecahedron
- who could invent the Monster if it did not exist?
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*A modern mathematical proof is not very different from a modern machine or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details.*

—Hermann Weyl

Step 1: (Feit–Thompson) Every (non-abelian) simple group has even order
Monstrous moonshine

- The Monster lives in a space with 196883 dimensions

- The \( j \)-function is the only 'simple' elliptic modular function (any other modular function can be written in terms of it)

\[
j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots \]

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The finite simple groups, 1

- rotation groups of the regular polygons which have a prime number of sides
- The *alternating groups* are rotation groups in $n$-dimensional space: rotation symmetries of $n + 1$ points equally spaced from each other. The reflection group permutes these points in all possible ways. The rotation group has exactly half the permutations, and is simple provided $n$ is at least 4.

The case $n = 5$:  

The finite simple groups, 2

- The *classical groups* can be described by ‘rolling up’ $n$-dimensional space into a finite space. In this rolling up process, every circle has to have the same prime number of points on it.

- The *exceptional groups* are related to the exceptional reflection groups. The most complicated groups in this family are the groups of type $E_8$: they need a space of 248 dimensions.

- The 26 *sporadic groups* are even more exceptional.
A group always arises in nature as the symmetry group of some object,

Given a group, we can ask for all possible objects that it can be the symmetry group of.

This is essentially what we call Representation theory.

Now combine the two processes:
1st object → symmetry group → 2nd object

Mathematicians do not study objects, but relations between objects. Thus they are free to replace some objects by others, so long as the relations remain unchanged.

—H. Poincaré
SOME EARLY WORK
Research strategies

▶ tackle easy problems

▶ tackle impossible problems because recognition comes only from solving hard problems, preferably ones which everyone else has given up on as they consider them essentially impossible.

Never discount the possibility of success!
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Bean-counting

One can measure the importance of a scientific work by the number of earlier publications made superfluous by it.

—David Hilbert

On this measure my most important paper is a little sideline which I wrote with John Conway and Peter Kleidman more than 20 years ago, in which we used reflection groups (specifically $E_8$) to construct lots of finite projective planes. Ernie Shult described our paper as “wiping out an entire area of research”, because we had convincingly demonstrated that there are just so many projective planes that a classification was essentially impossible.
Projective planes

- through every pair of points there is exactly one line, and
- every pair of lines meets in exactly one point.
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Maximal subgroups of sporadic simple groups

(this is equivalent to the problem of finding the simple permutation representations)

- the Suzuki group, of order 448,345,497,600
- the Rudvalis group, of order 145,926,144,000
- Conway’s second group, of order 42,305,421,312,000
- Conway’s first group, of order 4,157,776,806,543,360,000

Clearly an impossible problem.

— It took me three weeks.
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Simplify, simplify, simplify

*Everything should be made as simple as possible, but not simpler.*

—*Albert Einstein*
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Seek simplicity, and distrust it.

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The heart of my PhD was an observation so simple it is easy to miss:

If a group fixes a point in space, then it still fixes a point when the space is rolled up (modulo 2, in my case).
THE MONSTER
The Monster

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- The most compact form is in a space of 196882 dimensions over integers modulo 2.
- Each group element would require 5GB of space.
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Clearly impossible, when I tackled this problem in the early 1990s.
Constructing groups and representations

In June 1991 I made
   - the Harada–Norton group, in 133 dimensions
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- the Baby Monster, in 4370 dimensions

These matrices formed the start of what has now become the WWW Atlas of Group representations, which now has many thousands of representations of hundreds of groups.

http://brauer.maths.qmul.ac.uk/Atlas/v3/
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Taming the Monster?

- 5GB per matrix
- 3 months to multiply two matrices
- thousands of multiplications required
- by 1993 I was convinced that we (Richard Parker and I) had a plan that would work.
- my Ph.D. student Peter Walsh started work on it.
- I finished it off in the first half of 1997, with some help from Steve Linton.
The Monster tamed!

- **The good news:** not 5 gigabytes for each matrix, but about half a megabyte.

- **The bad news:** can only write down some of the symmetries: enough to generate the group.

- **More good news:** not three months to combine two symmetries, but a fraction of a second to apply a symmetry to a point.

  The whole matrix tells you what the symmetry does to every point. If you choose your point carefully enough, it is good enough to see where the symmetry takes one point.

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So the Monster was tamed.
The Monster reveals its secrets

- The Monster is a Hurwitz group
- A Hurwitz group is a bit like the rotation part of a reflection group, with mirrors at angles of $90^\circ$, $60^\circ$ and $25\frac{5}{7}^\circ$. This only makes sense in a ‘hyperbolic space’ in which triangles have their corners squashed in.

Pictures courtesy of Claudio Rocchini
Hurwitz geometry

Picture courtesy of Claudio Rocchini
More secrets of the Monster

- My student Beth Holmes completed another construction of the Monster which I had started. Using this, we found new maximal subgroups $L_2(59)$ and $L_2(29):2$.
- Her PhD thesis, and later work, nearly, but not quite, answered the whole maximal subgroup question.
- Moonshine and nets: more secrets to be revealed (ask Richard Barraclough).
SOME RECENT WORK
Ree groups

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- An example—the Ree groups. These are two of the ten families of exceptional simple groups.
- They were discovered in about 1960 by Rimhak Ree. One family lives in 7-dimensional space, the other lives in 26-dimensional space.
- Very soon afterwards, Jacques Tits gave a more geometrical construction of the 7-dimensional case.
Difficulty versus simplicity

That I have been able to accomplish anything in mathematics is really due to the fact that I have always found it so difficult. When I read, or when I am told about something, it nearly always seems so difficult, and practically impossible to understand, and then I cannot help wondering if it might not be simpler. And on several occasions it has turned out that it really was more simple!

—David Hilbert

This accurately describes my relationship with the Ree groups.
Building the Ree groups, 1

Start with the projective plane of order 2 and build a 7-dimensional space out of it: each blob represents one of the 7 dimensions, all perpendicular to each other. Now roll it up modulo 3, so that no matter what direction you walk in, after three steps you get back home.
Building the Ree groups, 2

- each line in the picture represents a (generalised) mirror—it fixes anything in the three directions it represents, and reflects (negates) everything in the other four perpendicular directions.
- the symmetry \( t \mapsto t + 1 \) rotates the 7 mirrors, each to the next.
- rotating the picture gives us the symmetry \( t \mapsto 2t \).
- Next, we build another 7-space according to the rule given in this picture.
Each arrow represents a wedge product, and the rule is

\[
t^* = (t + 1) \wedge (t + 3) - (t + 2) \wedge (t + 6) \\
(\text{mod} \ (t + 1) \wedge (t + 3) + (t + 2) \wedge (t + 6) \\
+ (t + 4) \wedge (t + 5))
\]

The Ree group is the group of symmetries which act the same in both 7-spaces simultaneously.

Actually, that is not true. There is a small technicality which is too technical to give in this lecture, but which is nevertheless relatively elementary.

There is a similar construction of the 26-dimensional Ree groups from a projective plane of order 3.
The future

_Young men should prove theorems, old men should write books._

—G. H. Hardy
The End

I hope I have convinced you of the exceptional simplicity of the exceptional simple groups.