MSM120—1M1
First year mathematics for civil engineers
Revision notes 4

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Series

A series is just an extended sum, where we may want to add up infinitely many numbers. In general it does not make sense to add up infinitely many things, so we have to be careful.

For example, a geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$$

can be continued indefinitely, but does the sum make sense? If $a = 1$ and $r = 2$, we get a sum $1 + 2 + 4 + 8 + \cdots$ which obviously "tends to infinity"—in the sense that it gets bigger and bigger indefinitely. In this case, we say the series *diverges*, or that the series does not have a sum (infinite numbers should never occur in engineering applications, or else something has gone seriously wrong!).

On the other hand if $r = \frac{1}{2}$, the sum is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Does this have a sum? Well, we can sum the series to n terms, and we get

$$\frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - 2.\left(\frac{1}{2}\right)^n$$

and as n gets larger, the last term tends to 0. (In other words, it gets so small that eventually we can ignore it.) So we can say that in the limit, as n tends to infinity, the sum tends to 2. We write this as

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2,$$

or in mathematical shorthand notation

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

Here the $\sum$ just stands for "the sum of", and the limits $n = 0$ and ∞ just mean we take all integer values of n from 0 upwards.

More generally, if

$$a_1 + a_2 + a_3 + \cdots$$

is a series, we write the "partial sums"

$$S_n = a_1 + a_2 + \cdots + a_n$$

of all the terms up to a_n, and write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

for the infinite sum if it exists.

The geometric series gives a model for how all series behave: if the common ratio r is between -1 and 1, then the series converges, and the sum is $\frac{a}{1-r}$, whereas if the common ratio is 1 or bigger, or -1 or smaller, then the series diverges. In this series, $a_n = ar^n$ and so $r = \frac{a_{n+1}}{a_n}$ for every positive integer n.

In general of course, a series doesn't have a common ratio, but it still has the ratios a_{n+1}/a_n, which may vary according to the value of n. If these ratios are *all* bigger than 1, then the series diverges. If they are all *significantly* less than 1, then the series converges: we need actually that the *limit as n tends to infinity* of a_{n+1}/a_n must be between -1 and 1, but not equal to -1 or 1. This is called D'Alembert's Ratio Test, and has its most important application to power series (see below).

An even more obvious test is the *divergence test* (sometimes called the *non-null test*): if the individual terms a_n do not tend to 0, then the series $\sum_{n=0}^{\infty} a_n$ cannot possibly converge.

On the other hand it is quite possible for the terms a_n to tend to 0, but for the series $\sum_{n=0}^{\infty} a_n$ to *diverge*. For example

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \end{aligned}$$

$$
\begin{aligned}
&= 1 + \frac{1}{2} + 2.\frac{1}{4} + 4.\frac{1}{8} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots
\end{aligned}
$$

which obviously diverges!

Power series

These are series where you introduce a variable x. They are of the form

$$\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$

You will come across many series like this as solutions to engineering problems, and the trick is to figure our how many terms you need to take in order to get a good enough approximation for engineering purposes. Of course, this will depend both on the series and on the application: generally in civil engineering you won't need many terms, but you do need to know how many terms you need to get an accurate enough answer.

In mathematics, however, we are much more demanding—we demand *infinitely many* terms.

The ratio test tells you when these series converge: you need $\lim b_{n+1}x/b_n$ to be between -1 and 1, which means you need x to be between $-\lim b_n/b_{n+1}$ and $\lim b_n/b_{n+1}$. Thus we call this latter number the *radius of convergence*, because if x is within this distance of 0, then the series converges, and if x is outside this distance form 0, then the series diverges.

There are many important examples, which describe functions we have already met. In addition to the binomial theorem

$$
\begin{aligned}
(1+x)^a &= \sum_{n=0}^{\infty} a(a-1)\cdots(a-n+1)\frac{x^n}{n!} \\
&= 1 + ax + a(a-1)\frac{x^2}{2} + a(a-1)(a-2)\frac{x^3}{6} + \cdots
\end{aligned}
$$

we have the following standard series which you should know:

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\
\cosh x &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} &&= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots \\
\sinh x &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} &&= x + \frac{x^3}{6} + \frac{x^5}{120} + \cdots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \\
\ln(1+x) &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x}{n} &&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots
\end{aligned}
$$

Limits and continuity

Consider the function $f(x) = \dfrac{\sin x}{x}$. This function is only defined when $x \neq 0$. When $x = 0$ it reduces to the "formula" $\dfrac{0}{0}$, which is meaningless. But as x gets very close to 0, the function is actually very well behaved. If we use the series expansion for $\sin x$, then we can see what happens as x gets very small:

$$
\begin{aligned}
\frac{\sin x}{x} &= \frac{x - x^3/3! + x^5/5! - \cdots}{x} \\
&= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots
\end{aligned}
$$

and as x gets smaller, all the terms except 1 become insignificant. And 'in the limit' as x approaches 0, the value of the function approaches 1.

We express this by writing

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

or: $\dfrac{\sin x}{x} \to 1$ as $x \to 0$. So one way to think of such a limit is as the value the function 'should' take if it behaved properly.

A more formal definition, as used by mathematicians to make things rigorous, is as follows. We say $\lim_{x \to a} f(x) = \ell$ if for every real number $\varepsilon > 0$ there is a real number $\delta > 0$ with the property that if $|x - a| < \delta$ then $|f(x) - \ell| < \varepsilon$.

(In this definition, ε measures how close you want the value of the function to get to ℓ, and δ measures how close you need x to be to a in order to make this approximation close enough.)

In general you can do algebra with limits in the same way you can with functions. The only problem is you must make sure you NEVER DIVIDE BY ZERO. So if $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$ then

$$\begin{aligned}
\lim_{x\to a} k.f(x) &= k.\ell \qquad \text{for any constant } k \\
\lim_{x\to a} (f+g)(x) &= \ell + m \\
\lim_{x\to a} (f.g)(x) &= \ell.m \\
\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{\ell}{m} \qquad \text{PROVIDED } m \neq 0
\end{aligned}$$

Sometimes you will find a function which tends to one limit as you approach $x = a$ from the left, and a different limit as you approach from the right. For example, if $f(x) = \dfrac{3}{4^{\frac{1}{x}} + 7}$ for $x \neq 0$, then when x tends to 0 through *positive* values, $\dfrac{1}{x}$ tends to $+\infty$, so $4^{\frac{1}{x}}$ tends to $+\infty$, and so $f(x)$ tends to 0. On the other hand, when x tends to 0 through *negative* values, $\dfrac{1}{x}$ tends to $-\infty$, so $4^{\frac{1}{x}}$ tends to 0, and so $f(x)$ tends to $\frac{3}{7}$. In this situation we talk about left and right limits, and we write

$$\begin{aligned}
\lim_{x\to 0+} f(x) &= 0 \\
\lim_{x\to 0-} f(x) &= \frac{3}{7}
\end{aligned}$$

Many, but by no means all, of the functions you will encounter will be *continuous*, which means that its graph can be drawn without taking your pencil off the page. More formally, we say that $f(x)$ is continuous at $x = a$, if $\lim_{x\to a} f(x) = f(a)$.

Examples: $\tan x$ is discontinous at $x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$, but continuous everywhere else.

The algebra of limits described above implies that if f and g are continuous functions, then so is kf (k a constant), $f+g$, $f.g$, and $f \circ g$ at the relevant points. Also f/g *except* where $g(x) = 0$.

Now you have probably used the fact that for a continuous function, if it starts off negative and ends up positive (or the other way round), then there must be a root in between. That is to say, if $f(x)$ is a continuous function on the interval $a \leq x \leq b$, and $f(a) < 0$ and $f(b) > 0$ (or $f(a) > 0$ and $f(b) < 0$), then there exists a number c with $a < c < b$ and $f(c) = 0$.

This is a form of the *Intermediate value theorem* (or IVT for short). It can be stated in a more general form: as the function goes from $f(a)$ to $f(b)$ it must go through *all values in between*.

Intermediate value theorem If $f(x)$ is a continuous function from the interval $a \leq x \leq b$ to $\mathbb{R}$, and $f(a) \neq f(b)$, then for every value d between $f(a)$ and $f(b)$, there is a number c between a and b such that $f(c) = d$.

Differentiation

If $f(x)$ is a function, we define the derivative

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

This is just the same as our earlier informal definition of $\frac{\mathrm{d}y}{\mathrm{d}x}$ as the limit of $\frac{\delta y}{\delta x}$ as δx tends to 0.

For example, if $f(x) = x^k$, for any real number k, then we use the binomial expansion of $(x + \delta x)^k$ as $(x^k + k.x^{k-1}\delta x + \cdots)$ to calculate

$$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \frac{(x^k + k.x^{k-1}\delta x + \cdots) - x^k}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{k.x^{k-1}\delta x + \cdots}{\delta x} \\ &= \lim_{\delta x \to 0} (k.x^{k-1} + \cdots) \\ &= k.x^{k-1} \end{aligned}$$

since all the remaining terms have a factor δx, so tend to 0.

Similarly, if $f(x) = e^x$, we use the power series expansion for e^x to get

$$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \frac{e^{x+\delta x} - e^x}{\delta x} \\ &= \lim_{\delta x \to 0} e^x \frac{e^{\delta x} - 1}{\delta x} \\ &= e^x . \lim_{\delta x \to 0} \frac{(1 + \delta x + \frac{(\delta x)^2}{2} + \cdots) - 1}{\delta x} \\ &= e^x . \lim_{\delta x \to 0} \frac{\delta x + \frac{(\delta x)^2}{2} + \cdots}{\delta x} \\ &= e^x . \lim_{\delta x \to 0} \left(1 + \frac{\delta x}{2} + \cdots\right) \\ &= e^x \end{aligned}$$

However, not all functions have derivatives, as the required limit may not exist. For example, let $f(x) = |x|$, and consider what happens near $x = 0$. We have

$$\frac{f(\delta x) - f(0)}{\delta x} = \frac{|\delta x|}{\delta x}$$

which is 1 if $\delta x > 0$, and -1 if $\delta x < 0$, so

$$\begin{aligned}
\lim_{\delta x \to 0+} \frac{f(\delta x) - f(0)}{\delta x} &= 1 \\
\lim_{\delta x \to 0-} \frac{f(\delta x) - f(0)}{\delta x} &= -1
\end{aligned}$$

which means the limit does not exist.

Differentiation of power series term by term. This always works, and the new power series has the same radius of convergence as the old one. For example, if $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$, then we differentiate term by term to get

$$\begin{aligned}
f'(x) &= 0 + 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots \\
&= e^x
\end{aligned}$$

To take another example:

$$\begin{aligned}
f(x) &= \sin x \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots \\
\text{so } f'(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!} + \cdots \\
&= \cos x
\end{aligned}$$

In the above calculations we have already used the basic rules of differentiation, namely

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(kf(x)) &= kf'(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}(f(x) + g(x)) &= f'(x) + g'(x)
\end{aligned}$$

You also already know the formula for differentiating a product

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x).g(x)) = f'(x).g(x) + f(x).g'(x)$$

We can now also derive the *Chain Rule* for differentiating a function of a function (i.e. a composite function). Suppose we have $y = f(x)$ and $z = g(y)$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta z}{\delta x}$$

$$
\begin{aligned}
&= \lim_{\delta x \to 0} \frac{\delta z}{\delta y} . \frac{\delta y}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\delta z}{\delta y} . \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\
&= \lim_{\delta y \to 0} \frac{\delta z}{\delta y} . \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\
&= \frac{\mathrm{d}z}{\mathrm{d}y} . \frac{\mathrm{d}y}{\mathrm{d}x}
\end{aligned}
$$

This can also be written in our other notation as

$$
\begin{aligned}
(g \circ f)'(x) &= g'(y).f'(x) \\
&= g'(f(x)).f'(x)
\end{aligned}
$$

For example, if $y = f(x) = 3x^2 + 1$ and $z = g(y) = \sqrt{y}$ then $z = \sqrt{3x^2 + 1}$, and we have

$$
\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{\mathrm{d}z}{\mathrm{d}y} . \frac{\mathrm{d}y}{\mathrm{d}x} \\
&= \frac{1}{2} y^{-1/2} . 6x \\
&= \frac{3x}{\sqrt{3x^2 + 1}}
\end{aligned}
$$

Now try an example yourself. Differentiate $e^{x^2 + 2\sin x}$. (You may want to put $y = x^2 + 2\sin x$ and $z = e^y$ so that you can follow the same method as above. With practice you will be able to write down the answer directly.)

Another consequence of the Chain Rule is the formula for differentiating a quotient: to differentiate $1/f(x)$, write $y = f(x)$ and $z = 1/y$, so that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{f(x)}\right) &= \frac{\mathrm{d}z}{\mathrm{d}y} . \frac{\mathrm{d}y}{\mathrm{d}x} \\
&= -\frac{1}{y^2} . f'(x) \\
&= \frac{-f'(x)}{(f(x))^2}
\end{aligned}
$$

and combining this with the formula for differentiating a product

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g(x)}{f(x)}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(g(x) . \frac{1}{f(x)}\right) \\
&= g(x) . \frac{-f'(x)}{(f(x))^2} + \frac{1}{f(x)} . g'(x) \\
&= \frac{f(x).g'(x) - g(x).f'(x)}{(f(x))^2}
\end{aligned}
$$

Implicit differentiation Often the relationship between two variables, such as x and y, is not so easily expressed in the form $y = f(x)$. Perhaps the relationship is only expressed in an *implicit* form, such as $x^2 + y^2 + xy = 1$. How do we determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ in such a situation? Well, we can differentiate the whole equation with respect to x, using the chain rule carefully, to get:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + y^2 + xy\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(0) \\
2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + (x\frac{\mathrm{d}y}{\mathrm{d}x} + 1.y) &= 0 \\
(2x + y) + (2y + x)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2x + y}{2y + x}
\end{aligned}$$

Of course, when we do this, the answer is expressed in terms of x and y, rather than just x.

Differentiation of inverse functions Inverse functions can be differentiated by using implicit differentiation as above. For example, if $y = \sin^{-1} x$ then $x = \sin y$, so $1 = \cos y.\frac{\mathrm{d}y}{\mathrm{d}x}$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec y$. Also, since $\sin y = x$, we have $\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}$, and looking at the graph of $\sin^{-1} x$ shows that the slope is always positive, so

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} x\right) = \sqrt{1 - x^2}$$

As an exercise, find the derivative of $y = \tan^{-1} x$.

Example If $x = e^y$, then $1 = e^y\frac{\mathrm{d}y}{\mathrm{d}x}$ and the inverse function is given by $y = \log_e x$, and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-y} = \frac{1}{x}.$$

This justifies our earlier assertion that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_e x) = \frac{1}{x}.$$

[However, it still depends on the power series exapnsion for e^x, which we have not justified.]

Logarithmic differentiation Certain functions, such as x^x, cannot be differentiated directly by the methods we have used so far. There is a trick, which is to take the logarithm first, and then use implicit differentiation. So, if $y = x^x$, then taking the log of both sides gives $\ln y = x \ln x$, and then differentiating both sides with respect to x gives

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x.\frac{1}{x} + 1.\ln x$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 + \ln x) = x^x(1 + \ln x).$$

This method works whenever you have a function of the form $y = f(x)^{g(x)}$. So for example, if $y = (\cos x)^{\sin x}$ then

$$\begin{aligned}
\ln y &= \sin x.(\ln \cos x) \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \sin x.\frac{1}{\cos x}.(-\sin x) + \cos x.(\ln \cos x) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= (\cos x)^{\sin x}.(-\sin x.\tan x + \cos x.(\ln \cos x))
\end{aligned}$$