MTH714U/MTHM024 Group Theory
Exercises 4: December 2009

Hints and solutions

1. *State and prove a version of Iwasawa's Lemma in which 'abelian' is replaced by 'soluble'.*

   If G is a finite perfect group, acting faithfully and primitively on a set Ω, such that the point stabilizer H has a normal soluble subgroup A whose conjugates generate G, then G is simple.

   Proof. For otherwise, there is a normal subgroup K with $1 < K < G$, which does not fix all the points of Ω, so we may choose a point stabilizer H with $K \not\leq H$. Therefore HK is a subgroup (since K is normal) which is strictly bigger then H, and so $G = HK$ since H is a maximal subgroup of G. So any $g \in G$ can be written $g = hk$ with $h \in H$ and $k \in K$, and therefore every conjugate of A is of the form $g^{-1}Ag = k^{-1}h^{-1}Ahk = k^{-1}Ak \leq AK$, since A is normal in H and K is normal in G. But G is generated by these A^k so $G = AK$. Therefore $G/K = AK/K \cong A/A \cap K$ is soluble, contradicting the assumption that G is perfect.

2. *(a) How many Sylow 3-subgroups does A_6 have? Show that A_6 acts 2-transitively on them, by conjugation.*

   (b) Apply Iwasawa's Lemma to this action to show that A_6 is simple.

   (a) 10. Each corresponds to a splitting of the 6 points into two 3s. The Sylow subgroup $\langle (1,2,3),(4,5,6) \rangle$ acts transitively on the other nine, so the action of A_6 is 2-transitive.

   (b) The stabilizer (i.e. normalizer) of a Sylow 3-subgroup has the Sylow 3-subgroup as a normal abelian subgroup. This contains 3-cycles, and the conjugates of these 3-cycles generate A_6. Moreover, these 3-cycles are commutators, so A_6 is perfect. Now apply Iwasawa's lemma.

3. *Compute the addition and multiplication tables for the fields*

   (a) $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2 + x + 1)$;

   (b) $\mathbb{F}_8 = \mathbb{F}_2[x]/(x^3 + x + 1)$;

   (c) $\mathbb{F}_9 = \mathbb{F}_3[x]/(x^2 + 1)$.

(a)

0	1	x	$x+1$
1	0	$x+1$	x
x	$x+1$	0	1
$x+1$	x	1	0

1	x	$x+1$
x	$x+1$	1
$x+1$	1	x

(b) The additional table is just addition of polynomials (of degree at most 2) mod 2. The multiplication table is as follows.

1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$

(c) This time I will write i for $\sqrt{-1}$. Addition is just modulo 3. The multiplication table is an 8×8 table, which I will simplify by only including one of a, $-a$, for each non-zero a in the field.

1	i	$1+i$	$1-i$
i	-1	$-1+i$	$1+i$
$1+i$	$-1+i$	$-i$	-1
$1-i$	$1+i$	-1	i

4. *Let $G = \mathrm{GL}_n(q)$. Prove that $Z(G) = \{\lambda I_n \mid 0 \neq \lambda \in \mathbb{F}_q\}$, where I_n is the $n \times n$ identity matrix.*

If the matrix (a_{ij}) commutes with $I + e_{kl}$, where e_{kl} is the matrix having a 1 in the (k, l) position and 0 elsewhere, then adding the kth row to the lth row of (a_{ij}) is equivalent to adding the lth column to the kth column. Therefore all the off-diagonal entries of (a_{ij}) are zero. Moreover, $a_{kk} = a_{ll}$.

5. *How many k-dimensional subspaces are there in a vector space of dimension n over the field of q elements?*

Choose k linearly independent vectors $v_1, \ldots, v_k$ (in order) in

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$$

ways. Now the space they span has a total of

$$(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})$$

different ordered bases, so the number of k-dimensional subspaces is the quotient of these, or, cancelling the factors of q,

$$\frac{q^n - 1}{q^k - 1} . \frac{q^{n-1} - 1}{q^{k-1} - 1} . \cdots . \frac{q^{n-k+1} - 1}{q - 1} .$$

6. *Prove, by induction on n or otherwise, that the characteristic polynomial of the matrix*

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & \\ \vdots & & \ddots & \ddots & \\ 0 & & & & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix}$$

is $x^n - a_{n-1}x^{n-1} - \dots - a_1 x - a_0$.

We calculate $\det(A - xI_n) = \begin{vmatrix} x & -1 & 0 & \dots & 0 \\ 0 & x & -1 & \dots & \\ \vdots & & \ddots & \ddots & \\ 0 & & & & -1 \\ -a_0 & -a_1 & -a_2 & \dots & x - a_{n-1} \end{vmatrix}$ as

$$\begin{aligned} & x \begin{vmatrix} x & -1 & \dots & \\ \vdots & \ddots & \ddots & \\ 0 & & & -1 \\ -a_1 & -a_2 & \dots & x - a_{n-1} \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & x & -1 & \dots & \\ \vdots & & \ddots & \ddots & \\ 0 & & & & -1 \\ -a_0 & -a_2 & -a_3 & \dots & x - a_{n-1} \end{vmatrix} \\ = \;\; & x^n - a_{n-1}x^{n-1} - \dots - a_1 x - a_0 \end{aligned}$$

using induction to evaulate the first determinant, and evaluating the second determinant directly, as there is only one non-zero term in the expansion.