MTH714U/MTHM024 Group Theory
Exercises 1: October 2009
Hints and solutions

1. *How many distinct composition series does the dihedral group D_8 have? Write them all down.*

   First note that all the composition factors have order 2. The best way to construct composition series is to start from the top and work down, not the other way round. Now D_8 has three subgroups of index 2: one is the subgroup C_4 of rotations, and the other two are $C_2 \times C_2$. If we label the vertices $1, 2, 3, 4$ in cyclic order, then these last two possibilities are $\{1, (1,3), (2,4), (1,3)(2,4)\}$ and $\{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$.

   Then C_4 has a unique composition series, while $C_2 \times C_2$ has three (since it has three subgroups of order 2).

   Thus there are 7 distinct composition series for D_8. I'll leave it to you to write them down explicitly.

2. *(a) Prove that the cyclic group C_n has a unique subgroup of each order d, where $d|n$.*

   (b) Write down all possible composition series for C_{60}.

   (c) If $n = {p_1}^{a_1}.{p_2}^{a_2}.\ldots.{p_r}^{a_r}$, where $p_1, \ldots, p_r$ are distinct primes, how many distinct composition series does C_n have?

   (d) How many distinct composition series does the dihedral group D_{2n} have?

   (a) Every subgroup of a cyclic group is cyclic, by Euclid's algorithm applied to the exponents of the generators g^i of the subgroup (where g is a generator of the original group). Now if g^i is the smallest power of g that is in the subgroup H, we see that $H = \{g^i, g^{2i}, \ldots\}$ so $i|n$ and H has order n/i.

   (b) There are four composition factors, C_2 (twice), C_3 and C_5. The only thing to consider is which order they come in: this can be any order, so we have 12 possibilities, such as

   $$1 < C_3 < C_6 < C_{30} < C_{60}$$

   etc etc.

   (c) This is a standard combinatorial problem: out of the $a_1 + a_2 + \cdots + a_r$ composition factors in order in the composition series, you have to choose a_1 places to put the factors C_{p_1} and so on. The answer is

   $$\frac{(a_1 + a_2 + \cdots + a_r)!}{a_1!a_2! \cdots a_r!}.$$

(d) If $G = D_{2n}$ with n odd, then the derived group $G' = C_n$, so every composition series goes through C_n, and the answer is the same as in (c).

If $G = D_{4n}$ then $G' = C_n$, and $G/G' \cong C_2 \times C_2$, so G has three subgroups of index 2. One of these is C_{2n} while the other two are D_{2n}. Now apply the same argument recursively and you get a horrible formula involving a sum of m terms, where $|G| = 2^m r$ with r odd.

3. *Let G be the subgroup of S_7 generated by the permutations $(1, 2, 3, 4, 5, 6, 7)$ and $(2, 6, 5, 7, 3, 4)$.*

(a) Find a normal subgroup N of G. To what well-known group is G/N isomorphic?

(b) Hence or otherwise find a composition series for G.

(c) Show that G has exactly seven conjugacy classes, and write down representatives for each of them.

(a) Let $x = (1, 2, 3, 4, 5, 6, 7)$ and $y = (2, 6, 5, 7, 3, 4)$. Then

$$x^y = (1, 6, 4, 2, 7, 5, 3) = x^5$$

so $\langle x\rangle$ is a normal subgroup, of order 7. The quotient $G/\langle x\rangle \cong C_6$, since it is generated by the coset $\langle x\rangle y$.

(b) Lifting the composition series $\langle x\rangle/\langle x\rangle < \langle\langle x\rangle y^3\rangle/\langle x\rangle < \langle\langle x\rangle y\rangle/\langle x\rangle$ to G we get the following composition series for G:

$$1 < \langle x\rangle < \langle x, y^3\rangle < G$$

The three composition factors are C_7, C_2 and C_3. Note that the group $\langle x, y^3\rangle \cong D_{14}$.

(c) G has order 42, and the normal subgroup of order 7 is the union of just two conjugacy classes, $\{1\}$ and $\{x, x^2, x^3, x^4, x^5, x^6\}$. Now x does not centralize any non-trivial element of $\langle y\rangle$, so each such element has at least 7 distinct conjugates. On the other hand each such element is centralized by $\langle y\rangle \cong C_6$, so by the orbit–stabilizer theorem has at most $42/6 = 7$ distinct conjugates. These five elements y, y^2, y^3, y^4, y^5 therefore represent five conjugacy classes of 7 elements each, accounting for all the 42 elements in G.