Queen Mary University of London

# MTH714U/MTHM024 Group theory

## Assignment 5 For handing in on 17 December 2009

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**1** Let $p$ be a prime, $p \geq 5$, and let $G = \mathrm{SL}_2(p)$. Show that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generates a Sylow $p$-subgroup $S$ of $G$.

Show that for any $\lambda \in \mathbb{F}_p \setminus \{0\}$ the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ is in $N_G(S)$.

What is the order of $N_G(S)$, and how many Sylow $p$-subgroups does $G$ have?

**2** Continuing the notation of the previous question, suppose that $p \equiv 1 \pmod{2^k}$, where $k \geq 2$, but $p \not\equiv 1 \pmod{2^{k+1}}$. Show that a Sylow 2-subgroup of $\mathrm{SL}_2(p)$ has order $2^{k+1}$ and is generated by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, where $\lambda$ is an element of order $2^k$ in $\mathbb{F}_p$.

Deduce that a Sylow 2-subgroup of $\mathrm{PSL}_2(p)$ is isomorphic to $D_{2^k}$.

**3** (a) List the conjugacy classes in $S_6$ with their cycle types, class sizes, and the centralizer of an arbitrary element in the class.

(b) For each class, give its image under the outer automorphism of $S_6$ (with proof).

**4** (a) Prove that two elements of $\mathrm{GL}_2(q)$ are conjugate if (and only if) they have the same characteristic polynomial, and they have the same minimal polynomial.

(b) The same for $\mathrm{GL}_3(q)$.

(c) Give an example to show that this is false for $\mathrm{GL}_4(q)$.

**5** Calculate the conjugacy classes in $\mathrm{GL}_2(3)$. For each class, give a representative of the class, its characteristic polynomial and minimal polynomial, its centralizer, and the size of the class.

Compare with the conjugacy classes in $\mathrm{PGL}_2(3) \cong S_4$. What do you notice?

**6** (Hard) Compute the conjugacy classes in $\mathrm{PGL}_2(9)$.

Compare with the conjugacy classes in $\mathrm{PSL}_2(9) \cong A_6$. State (and prove, if you can) a criterion for when a conjugacy class in $\mathrm{PGL}_2(9)$ (or more generally, $\mathrm{PGL}_2(q)$) splits into two classes in $\mathrm{PSL}_2(9)$ (or $\mathrm{PSL}_2(q)$).