# MTH714U/MTHM024 Group theory

## Assignment 3 For handing in on 19 November 2009

**1** Prove that if the permutation $\pi$ on $n$ points is the product of $k$ disjoint cycles (including trivial cycles), then $\pi$ is an even permutation if and only if $n-k$ is an even integer.

**2** Let $n$ be an odd positive integer, $n \geq 3$, and consider the action of the dihedral group $D_{2n}$ of order $2n$ on the $n$ vertices of a regular $n$-gon.

Decide (with proof) for which values of $n$ this action is primitive.

**3** Suppose that the finite group $G$ acts transitively on the set $\Omega$, and $|\Omega| = p$, where $p$ is prime.

Prove that $G$ acts primitively on $\Omega$.

**4** Let $G$ act transitively on $\Omega$. Show that the average number of fixed points of the elements of $G$ is 1, i.e.

$$\frac{1}{|G|}\sum_{g\in G} |\{x \in \Omega \mid x^g = x\}| = 1.$$

**5** (a) Determine the conjugacy classes of the alternating group $A_6$, and the sizes of these conjugacy classes.

(b) Deduce that $A_6$ is simple.

(c) Hence write down all the composition series of $S_6$.

**6** Let $G$ be the group of rotational symmetries of a regular dodecahedron. Prove that $G \cong A_5$.

[Hint: one method is to partition the 20 vertices into 5 sets of 4, each set forming the vertices of a regular tetrahedron.]

**7** Let $G$ be the group of permutations of 8 points $\{\infty, 0, 1, 2, 3, 4, 5, 6\}$ generated by $(0,1,2,3,4,5,6)$ and $(1,2,4)(3,6,5)$ and $(\infty,0)(1,6)(2,3)(4,5)$. Show that $G$ is 2-transitive. Show that the Sylow 7-subgroups of $G$ have order 7, and that their normalisers have order 21. Show that there are just 8 Sylow 7-subgroups, and deduce that $G$ has order 168. Show that $G$ is simple.