Queen Mary
University of London

B. Sc. Examination by course unit 2010

MTH6109 Combinatorics

Duration: 2 hours

Date and time: 15 December 2010, 1430h–1630h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): R. A. Wilson

Question 1 (a) Give, with explanation, an explicit formula for $\binom{n}{k}$. [4]

(b) By counting the number of subsets of a set of size n in two ways, show that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

[4]

(c) Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

[4]

Question 2 (a) How many solutions of the equation $x_1 + x_2 + x_3 + x_4 = 11$ are there in non-negative integers x_i? [3]

(b) How many solutions are there in positive integers x_i? [3]

(c) How many solutions have at least one of the $x_i = 0$? [3]

(d) How many solutions have exactly one of the $x_i = 0$? [3]

Question 3 Let $A = \{1, 2, 3, 4, 5, 6, 7\}$.

(a) How many sequences of length 4 can be constructed from the elements of A, if repetitions are not allowed? [3]

(b) How many sequences of length 4 can be constructed from the elements of A, if repetitions are allowed? [3]

(c) How many of the sequences in (a) contain exactly two odd numbers? [3]

(d) How many of the sequences in (a) contain at least two odd numbers? [3]

Question 4 Let (a_n) be the sequence of integers defined by the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

for $n \geq 2$, and the initial conditions $a_0 = 1$, $a_1 = 1$.

(a) Derive a formula for a_n, for all $n \geq 0$. [7]

(b) Use the recurrence relation to find the generating function for the sequence (a_n). [7]

Question 5 (a) Define the term *partition* of a set. [4]

(b) Define the Stirling numbers of the second kind $S(n, k)$. [3]

(c) Determine $S(3, k)$ for all $0 \leq k \leq 3$. [3]

(d) Write down a recurrence relation for $S(n, k)$ and use it to determine $S(4, 3)$. [5]

Question 6 (a) Explain what it means for two Latin squares to be *orthogonal*. [3]

(b) Use the integers modulo 3 to construct two orthogonal 3×3 Latin squares. [6]

Question 7 Let X be a finite set and let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ be a family of subsets of X.

(a) Explain what it means to say that $\mathcal{F}$ has a *system of distinct representatives*. [3]

(b) State the theorem of Hall giving a necessary and sufficient condition for $\mathcal{F}$ to have a system of distinct representatives. [4]

(c) Determine whether the following family of subsets of $\{a, b, c, d, e\}$ has a system of distinct representatives:

$$\mathcal{F} = \{A_1, A_2, A_3, A_4, A_5\}$$

where $A_1 = \{a, b, e\}$, $A_2 = \{a, c\}$, $A_3 = \{c, e\}$, $A_4 = \{a, b, d\}$, $A_5 = \{a, c, e\}$. [4]

Question 8 (a) State the *Principle of Inclusion and Exclusion*. [3]

(b) Define the term *derangement*. [2]

(c) Use the Principle of Inclusion and Exclusion to derive a formula for the number of derangements of n points. [7]

(d) Deduce that the number of derangements of n points is approximately $n!/e$. [3]

End of Paper