Section 4.7: Row space and column space

We've already come across some useful vector spaces associated with any particular $m \times n$ matrix A, such as

- the space spanned by the columns of A
- the nullspace $N(A)$ of A

Terminology 4.56 If $A \in \mathbb{R}^{m \times n}$

- the row space $\text{row}(A)$ is the vector space spanned by the rows of A. This is a subspace of $\mathbb{R}^{1 \times n}$
- the column space $\text{col}(A)$ is the vector space spanned by the columns of A. This is a subspace of $\mathbb{R}^{m \times 1}$

<u>Example 4.57</u> $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

- $\text{row}(A) = \{ \alpha(1,0,0) + \beta(0,1,0) \mid \alpha, \beta \in \mathbb{R} \}$
  $= \{ (\alpha, \beta, 0) \mid \alpha, \beta \in \mathbb{R} \}.$

  is a subspace of $\mathbb{R}^{1\times 3}$ of dimension 2

- $\text{col}(A) = \{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \}$
  $= \{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \}$

  is (a 2-dim. subspace of) $\mathbb{R}^{2\times 1}$

<u>Terminology 4.58</u> The <u>rank</u> of A (or, more precisely, the <u>row rank</u> of A) is: $\text{rank}(A) = \dim(\text{row}(A))$.

[The <u>column rank</u> of A is

$\text{colrank}(A) = \dim(\text{col}(A))$.]

[$\{(1,0,0), (0,1,0)\}$ is a <u>spanning set</u> (by definition), & is <u>linearly independent</u> because if $\alpha(1,0,0) + \beta(0,1,0) = (0,0,0)$ then $\alpha = 0$, and $\beta = 0$.]

Theorem 4.59 If A and B are row-equivalent matrices, then

$$\text{rank}(A) = \text{rank}(B).$$

Proof
- The rows of B are linear combinations of the rows of A.
  $\Rightarrow$ Every spanning vector of $\text{row}(B)$ is in $\text{row}(A)$.
  $\Rightarrow$ Every vector in $\text{row}(B)$ is in $\text{row}(A)$
  (because $\text{row}(A)$ is a vector space)
  $\Rightarrow$ $\text{row}(B) \subseteq \text{row}(A)$
- Similarly $\text{row}(A) \subseteq \text{row}(B)$.
- Hence $\text{row}(A) = \text{row}(B)$.
- In particular $\text{rank}(A) = \dim(\text{row}(A))$
  $= \dim(\text{row}(B)) = \text{rank}(B)$.

Done.

How to calculate the rank of a matrix?

- Use Gaussian ~~elim~~ algorithm (or Gauss-Jordan algorithm) to get the matrix A into (reduced) echelon form U.
- The non-zero rows of U are linearly independent.
- Therefore $\operatorname{rank}(A) = \operatorname{rank}(U)$ $=$ number of non-zero rows in U.

Example 4.60

Compute rank(A),

where $A = \begin{pmatrix} 1 & -3 & 2 \\ 1 & -2 & 1 \\ 2 & -5 & 3 \end{pmatrix}$

Solution: $A \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

$$\xrightarrow[R_3 - R_2]{} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

So row(A) = span$\{(1,-3,2), (0,1,-1)\}$

and has dimension 2.

$\Rightarrow$ rank$(A) = 2$.

Terminology 4.61 $\dim(N(A))$ is called the <u>nullity</u> of A, denoted $\mathrm{null}(A)$.

Example 4.62

What is the nullity of

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \ ?$$

(see example 4.16)

$$N(A) = \left\{ \alpha \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\} \quad \text{(3 free variables)}$$

$$\Rightarrow \dim(N(A)) = 3.$$

Remarks

- $\text{null}(A)$ = the number of <u>free variables</u> in the row echelon form of the linear system $Ax = 0$.
- $\text{rank}(A)$ = the number of non-zero rows in the echelon form of A
  = the number of <u>leading</u> variables.

Therefore:

<u>Theorem 4.63</u> If $A \in \mathbb{R}^{m \times n}$ then

$$\text{rank}(A) + \text{null}(A) = n.$$

<u>Proof</u> n = total no. of variables
= no. of leading variables + no. of free variables
= $\text{rank}(A) + \text{null}(A)$. <u>Done</u>.

This is called the $\boxed{\text{rank-nullity } \cancel{\text{formula}}}$ theorem.

We saw in examples that the row space and the column space of A have the same dimension.

Was ~~the~~ this a coincidence, or is it true in general?

Theorem 4.64 For any $A \in \mathbb{R}^{m \times n}$,

$$\dim(\mathrm{col}(A)) = \dim(\mathrm{row}(A)).$$

Proof Let U be the echelon form of A.

$\Rightarrow \mathrm{rank}(A) = \mathrm{rank}(U)$.

Let U_L = matrix consisting of the columns of U corresponding to leading variables

& A_L = ... of A

Now the columns of U_L are linearly independent

~~$\Rightarrow$ the columns of A_L are linearly independent~~

~~[illegible]~~

$\Rightarrow$

$\Rightarrow$ the equations $U_L x_L = 0$
has only the zero solution
(where x_L is the column vector of all
leading variables)

$\Rightarrow$ the equation $A_L x_L = 0$
has only the zero solution
(because A_L is row-equivalent to U_L)

$\Rightarrow$ the columns of A_L are linearly independent

$\Rightarrow$ $\dim(\mathrm{col}(A_L)) = \mathrm{rank}(A) = \dim(\mathrm{row}(A))$

$\Rightarrow$ $\dim(\mathrm{col}(A)) \geq \dim(\mathrm{row}(A))$.

- Now transpose A and do the same again:

$$\begin{aligned}\dim(\mathrm{row}(A)) &= \dim(\mathrm{col}(A^T)) \\ &\geq \dim(\mathrm{row}(A^T)) \\ &= \dim(\mathrm{col}(A)).\end{aligned}$$

- Putting these two together gives

$$\dim(\mathrm{row}(A)) = \dim(\mathrm{col}(A)).$$

Done.

Example (4.60 continued)

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 1 & -2 & 1 \\ 2 & -5 & 3 \end{pmatrix} \longrightarrow U = \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_L = \begin{pmatrix} 1 & -3 \\ 1 & -2 \\ 2 & -5 \end{pmatrix}, \quad x_L = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad U_L = \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$U_L x_L = 0 : \quad \left.\begin{aligned} x_1 - 3x_2 &= 0 \\ x_2 &= 0 \\ (\ 0 &= 0\ ) \end{aligned}\right\} \Rightarrow x_1 = x_2 = 0$$

$$A_L x_L = 0 : \quad \left.\begin{aligned} x_1 - 3x_2 &= 0 \\ x_1 - 2x_2 &= 0 \\ 2x_1 - 5x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 = x_2 = 0$$

i.e. $x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has only the trivial solution

i.e. $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ -5 \end{pmatrix}$ are linearly independent.

i.e. $\mathrm{col}(A) \supseteq \mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ -5 \end{pmatrix} \right) = \mathrm{col}(A_L)$

so has dimension ≥ 2

- Using the theorem we know now that

$$\mathrm{col}(A) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ -5 \end{pmatrix} \right)$$

i.e. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ -5 \end{pmatrix} \right\}$ form a basis for $\mathrm{col}(A)$.

This shows us <u>how to find</u> a basis for col(A) :

- bring A to row echelon form
- identify the leading variables
- the columns of A corresponding to the leading variables form a basis for col(A).