Linear Algebra I   Lecture with Dr. Bray

Example 3.14

Compute det A where $A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{pmatrix}$

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} \overset{R_4 + R_2}{=} \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

(expand down 1st column)

$$= -2 \times \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$$

$R_2 - 3R_1$:

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

Swap R_2 and R_3:

$$- \begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix}$$

Because it is upper triangular

$$-(1 \cdot (-3) \cdot 5) = 15$$

So $\det A = -2 \times 15 = -30$

Principle of Mathematical Induction

Let P be a property defined on $\mathbb{Z}$, or a subset of $\mathbb{Z}$ such as $\mathbb{N}^+$.

If there is an $a \in \mathbb{Z}$ (such as $a=1$) such that
(i) $P(a)$ holds and
(ii) $\forall\, b \geq a$, $P(b)$ holds implies that $P(b+1)$ holds

[i.e. if $P(b)$ holds then $P(b+1)$ holds]

Then $P(c)$ holds for all $c \geq a$, $c \in \mathbb{Z}$.

Thm 3.18
For all $n \times n$ matrices A we have $\det A = \det(A^T)$

PROOF
This proceeds by induction on n.
We verify the cases $n \leq 2$ directly.
(↑Exercise: Done in Geometry I)
Else we suppose this holds in the case $n=k$, and try to show it holds for the case $n=k+1$.
Let A be a $(k+1)\times(k+1)$ matrix.
Let $B = A^T$.
Thm 3.7 (Cofactor Expansion Thm) gives:

$$\det(A^T) = \det B = \sum_{j=1}^{k+1} (-1)^{1+j}\, b_{1j}\, B_{1j}$$

(where B_{1j} is B with row 1 and column j removed)

pto

B_{1j} is a $k \times k$ matrix, and so $\det B_{1j}^T =$

$\det B_{1j}$ by the inductive hypothesis.

But $B_{1j}^T = A_{j1}$ and $b_{1j} = a_{j1}$

So $\det A^T = \sum_{j=1}^{k+1} (-1)^{1+j} a_{j1} \det A_{j1}$

$= \det A$ (by cofactor expansion column 1)

($\det A^T$ was calculated as a cofactor expansion along row 1)

<u>Thm 3.21 / 3.11</u>

Let A be an $n \times n$ matrix and let E be an elementary matrix.
Then $\det(EA) = \det E \det A$

with $\det E = \begin{cases} -1 & \text{if } E \text{ has type I (row swap)}. \\ \alpha & \text{if } E \text{ has type II (scale a row by } \alpha) \\ 1 & \text{if } E \text{ has type III (add a multiple of one row to another)} \end{cases}$

(Thm 3.11 expressed this in terms of row operations, but these are equivalent to pre-multiplying a matrix by a suitable elementary matrix E.)

Thm 3.19

Let A be a square matrix.

If B is obtained from A by swapping two columns then $\det B = -\det A$.

If B is obtained from A by ~~swapping~~ multiplying a column by α $(\alpha \neq 0)$ then $\det B = \alpha \det A$

If B is obtained from A by adding λ times ~~by~~ one column to another then $\det B = \det A$.

PROOF

Immediate from 3.18 and 3.11/3.21.

Remark: The matrix B above has form AE for some suitable elementary matrix E.
(Exercise: work out details)

Proof of 3.21/3.11

Proof is by induction on n.

Cases $n \leq 2$ are done in Geometry I.

Now we suppose the result holds $\forall$ $k \times k$ matrices A, and try to establish the result when A is $(k+1) \times (k+1)$ $[k \geq 2]$

Let $B = EA$. Let i a row not involved in the row operation affected by E.

(A is at least 3×3 so such a row exists as E affects at most two rows)

Expand det A and det B along row i.

$$\det A = \sum_{j=1}^{k+1} (-1)^{i+j} a_{ij} \det A_{ij}$$

$$\text{and } \det B = \sum_{j=1}^{k+1} (-1)^{i+j} \det B_{ij}$$

Observe that $a_{ij} = b_{ij} \ \forall j$

Also B_{ij} is obtained from A_{ij} by performing the row operation corresponding to E, again for all j, and since A_{ij} / B_{ij} are $k \times k$ we have $\det B_{ij} = \lambda \det A_{ij}$ where $\lambda = \det E$ is $-1, \alpha, 1$ respectively in the three cases.

$$\therefore \det B = \sum_{j=1}^{k+1} (-1)^{i+j} a_{ij} \cdot \lambda \det A_{ij}$$

$$= \lambda \det A$$
$$= \det E \det A \qquad \square$$

Thm 3.15

A square matrix A is invertible $\Leftrightarrow$ $\det A \neq 0$.

PROOF

One can bring A into echelon form U using elementary row operations.

By Thm 3.11 / 3.21 each elementary row operation multiplies the determinant of A by something nonzero.

Therefore $\det U = \gamma \det A$, for some $\gamma \neq 0$.

We already know that A is invertible iff U only has 1's on its diagonal.
(U is upper triangular in any case)

So A is invertible $\Leftrightarrow$ $\det U = 1$, and this implies that $\det A = \frac{1}{\gamma} \neq 0$.

Else U has a 0 on the diagonal. But U is still upper triangular so $\det U = 0$.
Thus $\det A = 0$ as $\gamma \neq 0$. $\square$

Definition 3.16

A square matrix A is non-singular $\Leftrightarrow$ $\det A \neq 0$

Corollary 3.17

A matrix A is invertible $\Leftrightarrow$ A is non-singular

i.e., invertible = non-singular

Thm 3.22

Let B and A be $n \times n$ matrices.
Then $\det(BA) = \det B \det A$.

PROOF

Case I: B is not invertible.
Then BA is not invertible, as if it were, we should have $I = (BA)C$ for some matrix C, and this would give $B(AC) = I$.

So B has a one-sided inverse and Corollary 2.42 gives $(AC)B = I$, so that B is invertible.
contradiction

So (3.15) gives $\det B = \det BA = 0$.
So $\det BA = \det B \det A$.

Case II: B is invertible.
Inverse Matrix Thm (2.44) implies
$B = E_1 \dots E_r$ with E_i elementary matrices $\forall i$.
So by 3.21, writing $|A|$ for $\det A$, we get:

$$
\begin{aligned}
|BA| &= |E_1 E_2 \dots E_r B| \\
&= |E_1| \, |E_2 \dots E_r A| \\
&= |E_1| \, |E_2| \, |E_3 \dots E_r A| \\
&= \dots \\
&= |E_1| \, |E_2| \, |E_3| \dots |E_{r-1}| \, |E_r A| \\
&= |E_1| \, |E_2| \dots |E_r| \, |A|
\end{aligned}
$$

Substitute $A = I$ into the above. Get:

$$|B| = |BI| = |E_1| \, |E_2| \cdots |E_r| \quad \text{since } |I| = 1$$

Thus $\det(BA) = |BA| = |B||A| = \det(B)\det(A)$ as required

$\square$

Section 3.3 Cramer's Rule and a formula for A^{-1}

Let A be an $m \times n$ matrix.

Write $\underset{\sim}{a}_i \, (\in \mathbb{R}^m)$ for the i^{th} column of A.

i.e. vector a_i

where $1 \leq i \leq n$.

We shall thus write $A = (\underline{a}_1 \; \underline{a}_2 \cdots a_n)$ when we wish to emphasise the columns of A.

Similarly $B = (\underline{b}_1 \cdots \underline{b}_n)$

But $I_n = (\underline{e}_1 \; \underline{e}_2 \cdots \underline{e}_n)$ $\quad \left(\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ etc.} \right)$

If B is a $p \times m$ matrix, then the rules of matrix multiplication give us:

$$BA = (B\underline{a}_1 \; B\underline{a}_2 \cdots B\underline{a}_n)$$

In particular, if $A = I_m$, then $(\underline{b}_1 \cdots \underline{b}_m) = B$
$= BI = (B\underline{e}_1 \cdots B\underline{e}_m)$

Therefore $\underline{b}_i = B\underline{e}_i$ for all $\underline{e}_i$.

Now let A be an $n \times n$ matrix.

We consider the equation $A\underline{x} = \underline{b}$, for some $\underline{b} \in \mathbb{R}^n$

We know that $A\underline{x} = \underline{b}$ has a unique solution $\Leftrightarrow$ A is invertible.

From now on, we assume that A is invertible.
(Cramer's Rule does not apply in general)

<u>Notation:</u>

Given an $n \times n$ matrix A, and vector $\underline{b} \in \mathbb{R}^n$ we use $A_i(\underline{b})$ to denote the result of replacing the i^{th} column of A by $\underline{b}$.

Thus $A_i(\underline{b}) = (\underline{a}_1 \cdots \underline{a}_{i-1} \; \underline{b} \; \underline{a}_{i+1} \cdots \underline{a}_n)$

<u>Thm 3.23</u> (Cramer's Rule)

Let A be an invertible $n \times n$ matrix, and let $\underline{b} \in \mathbb{R}^n$.

Then the system of equations $A\underline{x} = \underline{b}$ has the unique solution given by:

$$x_i = \frac{\det A_i(\underline{b})}{\det A} \qquad \left(\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right)$$

<u>PROOF</u>

We have
$$\begin{aligned}
A I_i(\underline{x}) &= A(\underline{e}_1 \cdots \underline{e}_{i-1} \; \underline{x} \; \underline{e}_{i+1} \cdots \underline{e}_n) \\
&= (A\underline{e}_1 \cdots A\underline{e}_{i-1} \; A\underline{x} \; A\underline{e}_{i+1} \cdots A\underline{e}_n) \\
&= (\underline{a}_1 \cdots \underline{a}_{i-1} \; \underline{b} \; \underline{a}_{in} \cdots \underline{a}_n) \\
&= A_i(\underline{b}) \qquad \text{since } A\underline{e}_i = \underline{a}_i \ \forall i \text{ and } A\underline{x} = \underline{b}
\end{aligned}$$

Expanding along the ith row gives

$$\det I_i(\underline{x}) = x_i$$

$$\therefore \det A_i(\underline{b}) = \det(A\, I_i(\underline{x}))$$

$$= \det A \det(I_i(\underline{x})) \qquad \text{(by Thm 3.22)}$$

$$= x_i \det A$$

Since $\det A \neq 0$ get $x_i = \dfrac{\det A_i(\underline{b})}{\det A}$ $\square$

<u>Example</u> Consider $\left.\begin{array}{l} ax + by = e \\ cx + dy = f \end{array}\right\}$ $(ad - bc \neq 0)$

Matrix Form: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$

Cramer's rule gives $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \dfrac{ed - bf}{ad - bc}$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ce}{ad - bc}$$