Chapter 7: Eigenvalues & eigenvectors

Section 7.1 Definition and examples

<u>Definition</u> If $A \in \mathbb{R}^{n \times n}$, then $\underline{x} \in \mathbb{R}^n$ is an <u>eigenvector</u> of A, with <u>eigenvalue</u> λ if $A\underline{x} = \lambda \underline{x}$ (and $\underline{x} \neq \underline{0}$)

<u>Example</u> If $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$, $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

then $Au = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3u$

& $Aw = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2w$

So $\underline{u}$, $\underline{w}$ are eigenvectors of A.

Indeed $\{\underline{u}, \underline{w}\}$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

Writing the linear transformation $L: x \longmapsto Ax$ with respect to the basis $B = \{\underline{u}, \underline{w}\}$, the matrix

$$[L]_B^B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

$L(u) = 3\underline{u} + 0\underline{w}$

$L(w) = 0\underline{u} + 2\underline{w}$

How to find eigenvalues & eigenvectors?

If $A\underline{x} = \lambda \underline{x}$

then $A\underline{x} - \lambda \underline{x} = \underline{0}$

So $\underbrace{(A - \lambda I)}_{n \times n \text{ matrix}} \underline{x} = \underline{0}$

i.e. $\underline{x} \in N(A - \lambda I)$

If $\underline{x}$ is an eigenvector then $\underline{x} \neq \underline{0}$,

So $N(A - \lambda I)$ is non-zero

$\Leftrightarrow \underbrace{\det(A - \lambda I)}_{\text{polynomial of degree } n \text{ in } \lambda} = 0.$

called the **characteristic polynomial** of A.

So the eigenvalues of A are exactly the roots of the characteristic polynomial.

To summarize:

<u>Theorem 7.4</u> The following are equivalent:

(a) λ is an eigenvalue of A

(b) $(A - \lambda I)\underline{x} = \underline{0}$ has a non-zero solution

(c) $N(A - \lambda I) \neq \{\underline{0}\}$.

(d) $A - \lambda I$ is ~~non~~ singular

(e) $\det(A - \lambda I) = 0$.

<u>Terminology 7.5</u> $N(A - \lambda I)$ is called the <u>eigenspace</u> of A corresponding to λ.

<u>Example 7.6</u> $A = \begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix}$

$$\Rightarrow A - \lambda I = \begin{pmatrix} -7-\lambda & -6 \\ 9 & 8-\lambda \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \det(A - \lambda I) &= (-7-\lambda)(8-\lambda) - 9(-6) \\ &= \lambda^2 - \lambda - 56 + 54 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1) \end{aligned}$$

So the <u>characteristic equation</u> is

$$(\lambda - 2)(\lambda + 1) = 0.$$

$\Rightarrow$ the eigenvalues are $\lambda = 2$
and $\lambda = -1$.

<u>Case $\lambda=2$:</u> $A - \lambda I = \begin{pmatrix} -9 & -6 \\ 9 & 6 \end{pmatrix}$ is singular.

$(A-\lambda I)x = 0$ if & only if

$$9x_1 + 6x_2 = 0$$

So the eigenspace $= \left\{ \alpha \begin{pmatrix} 2 \\ -3 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$

& $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is an eigenvector for eigenvalue 2.

[$\alpha \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is also an eigenvector, when $\alpha \neq 0$.]

<u>Case $\lambda=-1$:</u> $A - \lambda I = \begin{pmatrix} -6 & -6 \\ 9 & 9 \end{pmatrix}$

$(A-\lambda I)x = 0$ if & only if

$$x_1 + x_2 = 0$$

$\Rightarrow$ eigenspace $= \left\{ \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$.

<u>Example 7.8</u> Find the eigenvalues of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{pmatrix}$$

$\Rightarrow \det(A-\lambda I) = (1-\lambda)(4-\lambda)(6-\lambda)$

$\Rightarrow$ eigenvalues of A are $1, 4, 6$.

More generally:

Theorem 7.9 The eigenvalues of a triangular matrix are the diagonal entries.

Theorem 7.10 (Change of basis for linear transformations)

If A and B are two $n \times n$ matrices, and $B = S^{-1}AS$ for some (invertible) S, then the characteristic polynomial of A is equal to the characteristic polynomial of B. Hence the eigenvalues of A are equal to the eigenvalues of B.

Proof Char. poly. of A is $\det(A-\lambda I)$ & char. poly. of B is $\det(B-\lambda I)$.

Now $(B-\lambda I) = S^{-1}AS - \lambda I$

$$= S^{-1}AS - S^{-1}(\lambda I)S$$

$$= S^{-1}(A-\lambda I)S.$$

$\Rightarrow \det(B-\lambda I) = \det\left(S^{-1}(A-\lambda I)S\right)$

$= \det(S^{-1})\det(A-\lambda I)\det S$

$= \det(A-\lambda I)$

Since $\det S^{-1} = (\det S)^{-1}$.

$\left(= \det(A-\lambda I) \cdot \underbrace{\det S^{-1} \cdot \det S} \right.$

$\left. = \det(A-\lambda I) \cdot \underbrace{\det(S^{-1}S)}_{=1.} \right)$

Done.

Section 7.2: Diagonalisation

"Diagonalisation" is a shorthand for "finding a basis consisting of eigenvectors" – we saw in the example at the beginning of this chapter that the matrix of a linear transformation with respect to a basis of eigenvectors is diagonal.

Diagonal matrices are very easy to calculate with.

Example 7.11 If $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

then $D^2 = DD = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}$

and more generally

$$D^k = DD^{k-1} = \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix}.$$

Definition 7.12

- Two $n \times n$ matrices A, B are called <u>similar</u> if there is an invertible matrix P such that

$$P^{-1}AP = B.$$

[in effect, they are the matrices of the <u>same</u> linear transformation, written with respect to (possibly) different bases.]

- An $n \times n$ matrix A is called <u>diagonalisable</u> if it is <u>similar</u> to a <u>diagonal</u> matrix.

<u>Consequence</u> If $P^{-1}AP = B$ then

$$\begin{aligned} B^2 &= (P^{-1}AP)(P^{-1}AP) \\ &= P^{-1}A(PP^{-1})AP = P^{-1}AIAP = P^{-1}AAP \\ &= P^{-1}A^2P \end{aligned}$$

and more generally

$$\begin{aligned} B^k &= (P^{-1}AP)(P^{-1}AP)\dots(P^{-1}AP) \\ &= P^{-1}A^kP \end{aligned}$$

<u>In particular</u> if D is diagonal and $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$

(A^k: hard to calculate; D^k: easy to calculate.)

Note If D is diagonal ($n \times n$) then the coordinate vectors $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$ are

eigenvectors of D

Example $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$$\Rightarrow D\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$D\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So the key to diagonalising matrices is finding the eigenvectors.

Helpful result 7.13

If $v_1, \dots, v_r$ are eigenvectors of A (n×n matrix), with distinct eigenvalues $\lambda_1, \dots, \lambda_r$, ($\lambda_i \neq \lambda_j$ if $i \neq j$)

then $v_1, \dots, v_r$ are linearly independent.

(so they span an r-dimensional subspace of $\mathbb{R}^n$.)

Proof If $c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_r \underline{v}_r = \underline{0}$.

we need to show all $c_i = 0$.

Now $A(c_1 \underline{v}_1 + \dots + c_r \underline{v}_r) = A\underline{0} = \underline{0}$.

$$\Rightarrow \lambda_1 c_1 \underline{v}_1 + \lambda_2 c_2 \underline{v}_2 + \dots + \lambda_r c_r \underline{v}_r = \underline{0}.$$

Also $\lambda_1 c_1 \underline{v}_1 + \lambda_1 c_2 \underline{v}_2 + \dots + \lambda_1 c_r \underline{v}_r = \underline{0}$.

$$\Rightarrow (\lambda_2 - \lambda_1) c_2 \underline{v}_2 + \dots + (\lambda_r - \lambda_1) c_r \underline{v}_r = \underline{0}.$$

We can keep going, eliminating one vector at a time, until we're left with

$$\underbrace{(\lambda_r - \lambda_{r-1}) \cdots (\lambda_r - \lambda_2)(\lambda_r - \lambda_1)}_{\neq 0} c_r \underline{v}_r = \underline{0}$$

$\Rightarrow c_r = 0$ since $\underline{v}_r \neq \underline{0}$

& $(\lambda_{r-1} - \lambda_{r-2}) \dots (\lambda_{r-1} - \lambda_2)(\lambda_{r-1} - \lambda_1) c_{r-1} \underline{v}_{r-1}$
$+ (\dots) c_r \underline{v}_r = \underline{0}$

$\Rightarrow c_{r-1} = 0$ & so on.

$\Rightarrow c_r = c_{r-1} = \dots = c_2 = c_1 = 0$. Done.

Main result 7.14

- An $n \times n$ matrix A is diagonalisable
  if and only if
  A has n linearly independent eigenvectors
- In more detail, $P^{-1}AP = D$, with D diagonal,
  if and only if
  the columns of P are n linearly independent eigenvectors of A.
  The diagonal entries of D are the corresponding eigenvalues.

Proof Suppose $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & & & \lambda_n \end{pmatrix}$

& $P = \left(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n \right)$ (column vectors $\underline{v}_1, \ldots \underline{v}_n$).

Then $PD = \left(\underline{v}_1 \cdots \underline{v}_n \right) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} =$

$$= \left(\lambda_1 \underline{v}_1, \lambda_2 \underline{v}_2, \ldots \lambda_n \underline{v}_n \right)$$

If $\underline{v}_1, \ldots \underline{v}_n$ are eigenvectors of A with eigenvalues $\lambda_1, \ldots \lambda_n$ then

$$AP = A \left(\underline{v}_1 \; \underline{v}_2 \ldots \underline{v}_n \right) = \left(\lambda_1 \underline{v}_1, \lambda_2 \underline{v}_2, \ldots \lambda_n \underline{v}_n \right)$$

$$\Rightarrow \quad AP = PD$$

Moreover P is invertible if & only if $\underline{v}_1, \ldots \underline{v}_n$

are linearly independent.

- If the columns of P are linearly independent eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_n$

then $AP = (\lambda_1 \underline{v}_1, \ldots \lambda_n \underline{v}_n)$

$$= P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

& P is invertible

$$\Rightarrow A = APP^{-1} = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1}.$$

$\Rightarrow A$ is diagonalisable.

- Conversely if A is diagonalisable, say

$$A = PDP^{-1} \quad \text{with } D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

then $AP = PD$

i.e. $A(\underline{v}_1, \ldots \underline{v}_n) = (\lambda_1 \underline{v}_1, \lambda_2 \underline{v}_2, \ldots \lambda_n \underline{v}_n)$

$\Rightarrow A\underline{v}_1 = \lambda_1 \underline{v}_1, \ldots \quad A\underline{v}_n = \lambda_n \underline{v}_n$

$\Rightarrow \underline{v}_1, \ldots \underline{v}_n$ are eigenvectors of A. with eigenvalues $\lambda_1, \ldots \lambda_n$

Also $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent since P is invertible.

Done.

Example 7.15 Diagonalise, if possible:

$$A = \begin{pmatrix} -7 & 3 & -3 \\ -9 & 5 & -3 \\ 9 & -3 & 5 \end{pmatrix}$$

Solution **Step 1** $\det(A - \lambda I) =$

$$\begin{vmatrix} -7-\lambda & 3 & -3 \\ -9 & 5-\lambda & -3 \\ 9 & -3 & 5-\lambda \end{vmatrix} = \dots$$

$$= -\lambda^3 + 3\lambda^2 - 4$$

$$= -(\lambda+1)(\lambda^2 - 4\lambda + 4)$$

$$= -(\lambda+1)(\lambda-2)^2$$

$\Rightarrow$ eigenvalues are -1 and 2.

Step 2 Find $N(A+I)$ and $N(A-2I)$ (i.e. the eigenspaces corresponding to the eigenvalues -1 and 2)

$$\begin{pmatrix} -6 & 3 & -3 \\ -9 & 6 & -3 \\ 9 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ if \& only if}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \text{Span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

& similarly $N(A-2I) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \right\}$.

Step 3

So if $P = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

then $P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Example 7.16 If $A = \begin{pmatrix} -6 & 3 & -2 \\ -7 & 5 & -1 \\ 8 & -3 & 4 \end{pmatrix}$

then $\det(A - \lambda I) = -(\lambda+1)(\lambda-2)^2$

$\Rightarrow$ eigenvalues are -1 and 2.

$N(A+I) = \text{Span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}$

$N(A-2I) = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right\}$

So in this case there are not enough eigenvectors to span the whole space, & therefore A is not diagonalisable.

More generally (7.17) If $A \in \mathbb{R}^{n\times n}$ and $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of A, then

A is diagonalisable

if & only if

$\dim N(A-\lambda_1 I) + \dots + \dim N(A-\lambda_k I) = n.$

Special case 7.18 If $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues, then A is diagonalisable.

Proof Each eigenvalue $\lambda_1, \dots, \lambda_n$ comes with at ~~on~~ least one non-zero eigenvector $\underline{v}_1, \dots, \underline{v}_n$.

(7.13) $\Rightarrow$ $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent.

$\Rightarrow$ they form a basis of $\mathbb{R}^n$

$\Rightarrow$ A is diagonalisable. Done.

Section 7:3 : Complex vector space

Everything we've done in Chapters 1–5 & 7 (BUT NOT Chapter 6) applies equally well if we extend the real scalars to complex scalars everywhere.

Example $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

So A has no real eigenvalues or eigenvectors, but if we allow complex numbers then

$$\lambda^2 + 1 = 0 \iff \lambda = \pm i$$

So we have two complex eigenvalues.

$\lambda = i$ $\quad N(A - iI) = N\left(\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\right)$

= set of solutions $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-ix_1 - x_2 = 0 \quad \leftarrow \times i =$$
$$x_1 - ix_2 = 0$$

Solutions: $x_2 = \alpha$, $x_1 = i\alpha$, so $\begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector.

Also $\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Similarly if $\lambda = -i$, we get an eigenvector

$$\begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \text{or} \quad i \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Section 7.4: Diagonalising symmetric matrices

There is a substantial theory of (real) symmetric matrices, which are important in applications to physics.

It turns out they can <u>always</u> be diagonalised. Moreover, the transition matrix P that does the diagonalisation can be chosen in a very special way — to be an <u>orthogonal matrix</u>, so that $P^{-1} = P^T$.

Let's do an example first:

<u>Example 7.31</u> $A = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{pmatrix}$

<u>Step 1</u> $\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5$

$= (1+\lambda)^2(5-\lambda)$

so the eigenvalues of A are -1 and 5.

<u>Step 2</u> $N(A+I)$ has a basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$

$N(A-5I)$ has a basis $\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$.

<u>Step 3</u> Use Gram-Schmidt to get an <u>orthogonal</u> basis for $N(A+I)$:

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

<u>Step 4</u> Normalize the basis vectors to have norm 1:

$$u_1 = \frac{1}{\sqrt{2}} v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{3}} v_2 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

& $u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

<u>Step 5</u> The transition matrix $P = (\underline{u}_1\ \underline{u}_2\ \underline{u}_3)$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Then P is orthogonal ($P^{-1} = P^T$)

& $P^{-1}AP = P^TAP$ is diagonal:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

with the eigenvalues on the diagonal. //

- Now let's try and sort out why all this works.

There are two main questions we have to answer:

1. Why are the eigenvectors (for distinct eigenvalues) <u>orthogonal</u> to each other?

2. Why are there <u>enough</u> eigenvectors to span the whole of $\mathbb{R}^n$?

Q.1 uses Chapter 6

Q.2 uses <u>complex</u> vectors.

Lemma 7.27 If $A \in \mathbb{R}^{n \times n}$ is symmetric (that is, $A^T = A$), and $x, y \in \mathbb{R}^n$, then

$$(A\underline{x}) \cdot \underline{y} = \underline{x} \cdot (A\underline{y})$$

Proof $(A\underline{x}) \cdot \underline{y} = (Ax)^T y$

$$= x^T A^T y$$

$$= x^T (Ay)$$

$$= \underline{x} \cdot (A\underline{y})$$ **Done.**

Consequence If $\underline{x}, \underline{y}$ are eigenvectors of A (symmetric) with eigenvalues $\lambda \neq \mu$, then $(A\underline{x}) \cdot \underline{y} = (\lambda \underline{x}) \cdot \underline{y} = \lambda (\underline{x} \cdot \underline{y})$

and $\underline{x} \cdot (A\underline{y}) = \underline{x} \cdot (\mu \underline{y}) = \mu (\underline{x} \cdot \underline{y})$

$\Rightarrow (\lambda - \mu)\, \underline{x} \cdot \underline{y} = 0$

$\Rightarrow \underline{x} \cdot \underline{y} = 0$ (since $\lambda - \mu \neq 0$).

This answers Q.1.

~~If A is a symmetric n × n real matrix,~~
~~we know that A~~

The fundamental theorem of algebra

Every (complex) polynomial (of degree n) has at least one complex root.

Say α is a root.

Then $(x - \alpha)$ is a factor of the polynomial.

$\Rightarrow$ by induction a polynomial of degree n factorises into n linear factors over $\mathbb{C}$.
(real or complex)

<u>Consequence</u> Every matrix has at least one <u>complex</u> eigenvalue.

Lemma 7.29 Every symmetric matrix $A \in \mathbb{R}^{n\times n}$ has at least one real eigenvalue, with corresponding real eigenvector $\underline{v} \in \mathbb{R}^n$.

Proof By the fundamental theorem of algebra, A has a complex eigenvalue $a+ib$ (with $a, b \in \mathbb{R}$) with a corresponding complex eigenvector $\underline{v} + i\underline{w}$ (with $\underline{v}, \underline{w} \in \mathbb{R}^n$).

This means $A(\underline{v}+i\underline{w}) = (a+ib)(\underline{v}+i\underline{w})$

$$\Rightarrow A\underline{v} + iA\underline{w} = (a\underline{v} - b\underline{w}) + i(b\underline{v} + a\underline{w})$$

So equating real and imaginary parts

$$A\underline{v} = a\underline{v} - b\underline{w}$$
$$A\underline{w} = b\underline{v} + a\underline{w}$$

$$\Rightarrow (A\underline{v}).\underline{w} = a\underline{v}.\underline{w} - b\underline{w}.\underline{w}$$
$$= \underline{v}.(A\underline{w}) = b\underline{v}.\underline{v} + a\underline{v}.\underline{w}$$

$$\Rightarrow -b\underline{w}.\underline{w} = b\underline{v}.\underline{v}$$
$$\Rightarrow b(\underline{v}.\underline{v} + \underline{w}.\underline{w}) = 0$$

But $\underline{v}.\underline{v} + \underline{w}.\underline{w} \neq 0$

$\Rightarrow$ $b = 0$

$\Rightarrow$ the eigenvalue $a + ib = a$ is real.

<u>Done</u>.

Actually, the proof shows more:

<u>Consequence</u> <u>Every</u> eigenvalue of a <u>symmetric</u> real matrix $A \in \mathbb{R}^{n \times n}$ is <u>real</u>.

Now we need to show that the eigenvectors span the whole space.

<u>If not</u> we can use Gram-Schmidt to get an <u>orthogonal</u> set of eigenvectors $\underline{v}_1, \ldots, \underline{v}_k$, which span the same space as <u>all</u> the eigenvectors.

(Call this space H.) (Want to show $H = \mathbb{R}^n$.)

If we have run out of eigenvectors at this stage, then we know there are <u>no</u> eigenvectors in $H^\perp$.

(We want to show $H^\perp = \{0\}$.)

<u>Lemma</u> If $\underline{x} \in H^\perp$ then $A\underline{x} \in H^\perp$

<u>Proof</u> $\underline{x} \in H^\perp \Rightarrow \underline{x} \cdot \underline{v}_i = 0$ for $i = 0, \ldots, k$

$$\Rightarrow A\underline{x} \cdot \underline{v}_i = \underline{x} \cdot (A\underline{v}_i) = \underline{x} \cdot \lambda_i \underline{v}_i$$
$$= \lambda_i (\underline{x} \cdot \underline{v}_i) = 0 \quad \text{for all } i$$

$$\Rightarrow A\underline{x} \in H^\perp.$$

Done.

Now let $\underline{w}_1, \dots, \underline{w}_r$ be an orthonormal basis for $H^\perp$ & let B be the matrix of the linear transformation $\underline{x} \longmapsto A\underline{x}$ with respect to the basis $\underline{w}_1, \dots, \underline{w}_r$.

<u>Claim</u> B is symmetric

<u>Proof</u>

$$
\begin{aligned}
b_{ij} &= \text{coefficient of } \underline{w}_j \text{ in } A\underline{w}_i \\
&= (A\underline{w}_i) \cdot \underline{w}_j \\
&= \underline{w}_i \cdot (A\underline{w}_j) \\
&= b_{ji}.
\end{aligned}
$$

Done.

Hence B has an eigenvector $\in H^\perp$

Contradiction.

<u>This answers Q. 2</u>