Chapter 4: Vector spaces

The point of defining vector spaces in the abstract is so that the methods of linear equations and matrices can be used in a much more general context.

We saw that column vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

are $n \times 1$ matrices, which we can

- add: $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$

- multiply by scalars: $\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$

and these operations satisfied lots of conditions (Theorem 2.7)

Section 4.1: Definition and examples of vector spaces

Terminology A *vector space* is a non-empty set, V, with operations of *addition* and *scalar multiplication*, such that the properties in Theorem 2.7 hold, that is (for all $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$ and all $\alpha, \beta \in \mathbb{R}$)

- **Closure**: (C1) $\underset{\sim}{u} + \underset{\sim}{v} \in V$.
  (C2) $\alpha \underset{\sim}{u} \in V$.
- **Addition**: (A1) $u + v = v + u$ (commutative)
  (A2) $u + (v + w) = (u + v) + w$ (associative)
- **Zero**: (A3) there is $\underset{\sim}{0} \in V$, such that $\underset{\sim}{u} + \underset{\sim}{0} = \underset{\sim}{u}$.
  (A4) for every $\underset{\sim}{u} \in V$ there is an element $-\underset{\sim}{u} \in V$ such that $\underset{\sim}{u} + (-\underset{\sim}{u}) = \underset{\sim}{0}$.

(In particular V is an abelian group with the operation $+$).

- Distributive laws: (A5) $\alpha(u+v) = \alpha u + \alpha v$

  (A6) $(\alpha+\beta)u = \alpha u + \beta u$

- Scalars: (A7) $(\alpha\beta)u = \alpha(\beta u)$

  (A8) $1.u = u$

The elements $u \in V$ are called vectors.

Example 4.2 The set $\mathbb{R}^{m\times n}$ of all $m \times n$ matrices is a vector space. (Theorem 2.7)

Example 4.3 Let P_n denote the set of all **polynomials** of degree $\leq n$.

That is, each $\underset{\sim}{p} \in P_n$ is a polynomial

$$\underset{\sim}{p}(t) = a_0 + a_1 t + \cdots + a_n t^n .$$

Define **scalar multiplication** by

$$(\alpha \underset{\sim}{p})(t) = \alpha a_0 + \alpha a_1 t + \cdots + \alpha a_n t^n$$

and **addition** by:

if $\underset{\sim}{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$

then $(\underset{\sim}{p} + \underset{\sim}{q})(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n$

Then P_n is a vector space.

Zero polynomial $\underset{\sim}{0}(t) = 0 + 0.t + 0.t^2 + \cdots + 0.t^n$

$$\Rightarrow (\underset{\sim}{p} + \underset{\sim}{0})(t) = (a_0 + 0) + (a_1 + 0)t + \cdots = \underset{\sim}{p}(t).$$

$$-\underset{\sim}{p}(t) = -a_0 - a_1 t - \cdots - a_n t^n .$$

Example 4.4 Let $C[a,b]$ be the set of continuous real-valued functions on the interval $[a,b]$.

Define $(f+g)(t) = f(t) + g(t)$

and $(\alpha f)(t) = \alpha . f(t)$.

Then $C[a,b]$ is a vector space.

Is this obvious?

If not: (C1) $f+g$ ~~is a cont~~ $\in C[a,b]$

(C2) $\alpha f \in C[a,b]$

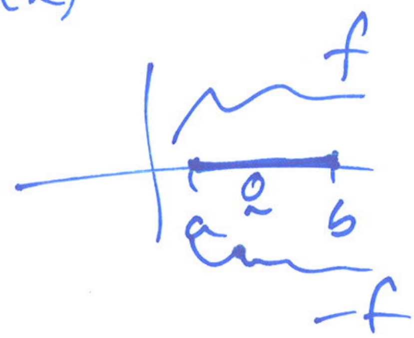
(A1) $(f+g)(t) = f(t) + g(t)$
$= g(t) + f(t)$
$= (g+f)(t)$

i.e. $f+g = g+f$ as functions.

(A2) $(f+g)+h = f+(g+h)$

(A3) $\underset{\sim}{0}(t) = 0$

$\underset{\sim}{0} + \underset{\sim}{f} = \underset{\sim}{f}$.



(A4) $(-f)(t) = -f(t)$.

(A5) $\alpha(f+g) = \alpha f + \alpha g$

because $[\alpha(f+g)](t) = \alpha[(f+g)(t)]$

$= \alpha(f(t)+g(t))$

$= \alpha f(t) + \alpha(g(t))$ by distributive law in $\mathbb{R}$

$= (\alpha f)(t) + (\alpha g)(t)$

$= (\alpha f + \alpha g)(t)$

(A6) $(\alpha+\beta)f = \alpha f + \beta f$

(A7) $\alpha(\beta f) = (\alpha\beta) f$

(A8) $1.f = f$

More facts (4.5) about vectors

- $0\underset{\sim}{u} = \underset{\sim}{0}$
- if $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0}$ then $\underset{\sim}{v} = -\underset{\sim}{u}$
- $(-1)\underset{\sim}{u} = -\underset{\sim}{u}$

Reasons

- $$\begin{aligned}\underset{\sim}{u} &= 1\underset{\sim}{u} \\ &= (1+0)\underset{\sim}{u} \\ &= 1\underset{\sim}{u} + 0\underset{\sim}{u} \\ &= \underset{\sim}{u} + 0\underset{\sim}{u}\end{aligned}$$

  $$\Rightarrow -\underset{\sim}{u} + \underset{\sim}{u} = -\underset{\sim}{u} + \underset{\sim}{u} + 0\underset{\sim}{u}$$

  $$\Rightarrow \underline{\underset{\sim}{0} = \underset{\sim}{0} + 0\underset{\sim}{u} = 0\underset{\sim}{u}}$$

- if $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0}$

  then $-\underset{\sim}{u} + \underset{\sim}{u} + \underset{\sim}{v} = -\underset{\sim}{u} + \underset{\sim}{0}$

  $$\Rightarrow \underset{\sim}{0} + \underset{\sim}{v} = -\underset{\sim}{u}$$

  $$\Rightarrow \underline{\underset{\sim}{v} = -\underset{\sim}{u}}$$

- $$\begin{aligned}\underset{\sim}{0} &= 0\underset{\sim}{u} \\ &= (1+(-1))\underset{\sim}{u} \\ &= 1\underset{\sim}{u} + (-1)\underset{\sim}{u} \\ &= \underset{\sim}{u} + (-1)\underset{\sim}{u}\end{aligned}$$

  & the result follows from the second result above.

Section 4.2: Subspaces

A subspace is "really" the set of solutions to a <u>homogeneous</u> linear system.

If x and y are two solutions to the same homogeneous linear system, say

$$A x = \underset{\sim}{0}$$

and $A y = \underset{\sim}{0}$,

then $A(x+y) = \underset{\sim}{0}$

and $A(\alpha x) = \underset{\sim}{0}$ for any scalar α.

We take these properties as the definition:

<u>Terminology</u> A <u>subspace</u> of a vector space V is any non-empty subset H satisfying:

- if $u \in H$ and $v \in H$ then $u + v \in H$;
- if $u \in H$ and α is a scalar, then $\alpha u \in H$.

Fact 4.7 If H is a subspace of V, then H is itself a vector space, with the same operations of addition and scalar multiplication as V.

Reasons Isn't it obvious?
(C1) + (C2) are from the defn. of subspace.
(A1)–(A8) are restrictions of (A1)–(A8) for V, to vectors $u, v, w \in H \subseteq V$.

Trivial examples (4.8)

- V is a subspace of itself.
- $\{\underset{\sim}{0}\}$ is a subspace of V.

because $\underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0} \in \{\underset{\sim}{0}\}$

and $\alpha \underset{\sim}{0} = \underset{\sim}{0}$

[because $\underset{\sim}{0} = 0 \cdot \underset{\sim}{u}$ for every $\underset{\sim}{u} \in V$

$= (\alpha 0) \cdot \underset{\sim}{u}$

$= \alpha (0 \underset{\sim}{u})$

$= \alpha \underset{\sim}{0}$.]

Example 4.9 The following are subspaces of $\mathbb{R}^3$:

- $L = \{ (r,s,t)^T \mid r,s,t \in \mathbb{R} \text{ and } r=s=t \}$
- $P = \{ (r,s,t)^T \mid r,s,t \in \mathbb{R} \text{ and } r-s+3t=0 \}$

Why? • $L \neq \emptyset$ because $(0,0,0)^T \in L$

• if $(r,s,t)^T \in L$
and $(a,b,c)^T \in L$

then $r=s=t$ and $a=b=c$

so $r+a = s+b = t+c$

$\Rightarrow (r+a, s+b, t+c)^T \in L$

$\Rightarrow (r,s,t)^T + (a,b,c)^T \in L$

• if $(r,s,t)^T \in L$ and $\alpha \in \mathbb{R}$

then $r=s=t \Rightarrow \alpha r = \alpha s = \alpha t$

$\Rightarrow \alpha (r,s,t)^T \in L.$

Lecture 15

Example 4.11

$$H = \{ (r^2, s, r)^T \mid r, s \in \mathbb{R} \}.$$

Is H a subspace of $\mathbb{R}^3$?

No, because $(1, 0, 1) \in H$

but $3(1, 0, 1) \notin H$

Example 4.12

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

Is H a subspace of $\mathbb{R}^{2\times 2}$?

No, because $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin H$.

(Or, because $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in H$

but $(-1)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \notin H$.)

Example 4.13

$$H = \{ \underset{\sim}{f} \in C[-2,2] \mid \underset{\sim}{f}(1) = 0 \}.$$

Is H a subspace of $C[-2,2]$?

Yes : • $\underset{\sim}{0} \in H$.

• If $\underset{\sim}{f}, \underset{\sim}{g} \in H$ then $\underset{\sim}{f}(1) = 0$ and $\underset{\sim}{g}(1) = 0$

$\Rightarrow (\underset{\sim}{f} + \underset{\sim}{g})(1) = \underset{\sim}{f}(1) + \underset{\sim}{g}(1)$
$= 0 + 0 = 0.$

$\Rightarrow \underset{\sim}{f} + \underset{\sim}{g} \in H.$

• If $\underset{\sim}{f} \in H$ and $\alpha \in \mathbb{R}$,

then $\underset{\sim}{f}(1) = 0$, $\Rightarrow (\alpha \underset{\sim}{f})(1) = \alpha \underset{\sim}{f}(1)$
$= \alpha . 0 = 0.$

$\Rightarrow \alpha \underset{\sim}{f} \in H.$

__Terminology 4.14__ If $A \in \mathbb{R}^{m \times n}$ the set of solutions to $Ax = 0$ is called the __nullspace of A__, written

$$N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

__Fact 4.15__ $N(A)$ is a subspace of $\mathbb{R}^n$.

__Why?__ • $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in N(A) \Rightarrow N(A) \neq \emptyset$

• If $x, y \in N(A)$ then

$Ax = 0$ and $Ay = 0$

$\Rightarrow A(x+y) = 0 \Rightarrow x + y \in N(A)$.

• If $x \in N(A)$ and $\alpha \in \mathbb{R}$

then $A(\alpha x) = \alpha (Ax) = \alpha 0 = 0$.

$\Rightarrow \alpha x \in N(A)$.

Section 4.3 : The span of a set of vectors

Terminology 4.17

If $v_1, \dots, v_n \in V$ the span of $v_1, \dots, v_n$ is the set of all linear combinations of $v_1, \dots, v_n$.

That is,

$$\mathrm{Span}(v_1, \dots, v_n) = \{\alpha_1 v_1 + \cdots + \alpha_n v_n \mid \alpha_1, \dots, \alpha_n \in \mathbb{R}\}$$

Example 4.18

If $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, in $\mathbb{R}^3$,

then
$$\mathrm{Span}(e_1, e_2) = \left\{ \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} \;\middle|\; \alpha_1, \alpha_2 \in \mathbb{R} \right\}.$$
$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; z = 0 \right\}$$
is a subspace.

If also $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

then $\mathrm{Span}(e_1, e_2, e_3) = \mathbb{R}^3$.

Example 4.19 If $v_1, v_2 \in V$, then $\mathrm{Span}(v_1, v_2)$ is a subspace of V.

Why?

- $v_1 = 1.v_1 + 0.v_2 \in \mathrm{Span}(v_1, v_2)$

  So it's non-empty.

- if $u, w \in \mathrm{Span}(v_1, v_2)$

  then $u = \alpha_1 v_1 + \alpha_2 v_2$ (some $\alpha_1, \alpha_2 \in \mathbb{R}$)

  and $w = \beta_1 v_1 + \beta_2 v_2$ (some $\beta_1, \beta_2 \in \mathbb{R}$)

  $\Rightarrow u + w = (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2$

  $\in \mathrm{Span}(v_1, v_2)$

- Similarly, $\alpha w = (\alpha \beta_1) v_1 + (\alpha \beta_2) v_2$

  $\in \mathrm{Span}(v_1, v_2)$.

Fact 4.20 If $v_1, \dots, v_n \in V$ then $\mathrm{Span}(v_1, \dots, v_n)$ is a subspace of V.

Why?

Essentially the same reasons.

Special case 4.21

If $\mathrm{Span}(v_1, \dots, v_n) = V$ we say $\{v_1, \dots, v_n\}$ is a <u>spanning set</u> for V.

In othe words: every vector in V is a linear combination of $v_1, \dots, v_n$

Example 4.22

Which of the following are spanning sets of $\mathbb{R}^3$:

(a) • $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right\}$.

(b) • $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

(c) • $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(d) • $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \right\}$.

(a) <u>Yes</u>: every $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 0.\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

(b) Given $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, want x, y, z such that

$$x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

i.e.
$$\begin{aligned} x+y+z &= a \\ x+y &= b \\ x &= c \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} z &= a-b \\ y &= b-c \end{aligned}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (b-c)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (a-b)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(c) $$\text{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 \end{pmatrix} \middle| \alpha_1, \alpha_2 \in \mathbb{R} \right\} \not\ni \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

So this is not a spanning set.

(d) $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + z\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$

$$\begin{aligned} x + 2y + 4z &= a \\ 2x + y - z &= b \\ 4x + 3y + z &= c \end{aligned}$$

Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 2 & 1 & -1 & b \\ 4 & 3 & 1 & c \end{array}\right) \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & -3 & -9 & b-2a \\ 0 & -5 & -15 & c-4a \end{array}\right) \xrightarrow{-\frac{1}{3}R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & 1 & 3 & -(b-2a)/3 \\ 0 & -5 & -15 & c-4a \end{array}\right) \xrightarrow[R_3+5R_2]{}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & 1 & 3 & -(b-2a)/3 \\ 0 & 0 & 0 & \underline{c-4a-5(b-2a)/3} \end{array}\right)$$

which is inconsistent for <u>some</u> vector

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, e.g. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\Rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \notin \mathrm{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \right)$.

$\Rightarrow \mathrm{Span}(\ldots) \neq \mathbb{R}^3$.

Example 4.23

Let $p_1(x) = 2 + 3x + x^2$

$p_2(x) = 4 - x$

$p_3(x) = -1$

be polynomials in $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$.

Show that p_1, p_2, p_3 span P_2.

Solution (Exercise.)

Lecture 16

A <u>spanning set</u> for a vector space gives us an economical way of describing the space.

An <u>important example</u> of a vector space is the set of solutions to a <u>homogeneous linear system</u>.

We also described this as the <u>nullspace</u> of the coefficient matrix.

Example 4.24

Find a spanning set for $N(A)$, where $A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$

Step 1 Put A into row echelon form

$$\xrightarrow[R_3-2R_2]{R_1+3R_2} \begin{pmatrix} 0 & 0 & 5 & 10 & -10 \\ 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \end{pmatrix}$$

(not quite the Gaussian algorithm!)

$$\xrightarrow{R_1-5R_3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \end{pmatrix}$$

So the leading variables are x_1 and x_3

the free variables are x_2, x_4, x_5

Step 2 Write down the general solution

$$\left.\begin{aligned} x_2 &= \alpha \\ x_4 &= \beta \\ x_5 &= \gamma \end{aligned}\right\} \Rightarrow \begin{aligned} x_3 &= -2\beta + 2\gamma \\ x_1 &= 2\alpha - 2x_3 - 3\beta + \gamma \\ &= 2\alpha + \beta - 3\gamma \end{aligned}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2\alpha+\beta-3\gamma \\ \alpha \\ -2\beta+2\gamma \\ \beta \\ \gamma \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

Step 3 Hence $\left\{ \begin{pmatrix} 2\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\-2\\1\\0 \end{pmatrix}, \begin{pmatrix} -3\\0\\2\\0\\1 \end{pmatrix} \right\}$

is a spanning set for $N(A)$.

[$N(A) = \left\{ \alpha \begin{pmatrix} 2\\1\\0\\0\\0 \end{pmatrix} + \beta \begin{pmatrix} 1\\0\\-2\\1\\0 \end{pmatrix} + \gamma \begin{pmatrix} -3\\0\\2\\0\\1 \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma \in \mathbb{R} \right\}$.]

Section 4.4: Linear independence

The above example shows that, in reality, we don't want any old spanning set, we want a minimal spanning set, i.e. containing as few vectors as possible.

The concepts of linear dependence and independence enable us to tell when we have a minimal spanning set.

Example If $x_1 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} -5 \\ 1 \\ 8 \end{pmatrix}$

then $x_3 = 3x_1 + 2x_2$.

Therefore $\mathrm{Span}(x_1, x_2, x_3) =$

$\{ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \}$

$= \{ \alpha_1 x_1 + \alpha_2 x_2 + 3\alpha_3 x_1 + 2\alpha_3 x_2 \}$

$= \{ (\alpha_1 + 3\alpha_3) x_1 + (\alpha_2 + 2\alpha_3) x_2 \}$

$\subseteq \mathrm{Span}(x_1, x_2)$.

But obviously $\mathrm{Span}(x_1, x_2) \subseteq \mathrm{Span}(x_1, x_2, x_3)$

so $\mathrm{Span}(x_1, x_2, x_3) = \mathrm{Span}(x_1, x_2)$.

Similarly $x_2 = -\frac{3}{2} x_1 + \frac{1}{2} x_3$

$\Rightarrow \mathrm{Span}(x_1, x_2, x_3) = \mathrm{Span}(x_1, x_3)$

and $x_1 = -\frac{2}{3} x_2 + \frac{1}{3} x_3$

$\Rightarrow \mathrm{Span}(x_1, x_2, x_3) = \mathrm{Span}(x_2, x_3)$.

On the other hand, there is no redundancy in the spanning set $\{x_1, x_2\} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$

Why not?

Because if $\alpha x_1 + \beta x_2 = 0$ then

$$\alpha \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.
$$\begin{aligned} -3\alpha + 2\beta &= 0 \\ \alpha - \beta &= 0 \\ 2\alpha + \beta &= 0 \end{aligned}$$

$$\Rightarrow \quad \alpha = \beta = 0$$

In other words there is no nontrivial relation of the form $\alpha x_1 + \beta x_2 = 0$.

This leads us to the general definition:

<u>Terminology 4.25</u> The vectors $v_1, v_2, \ldots, v_n$ in a vector space V are <u>linearly dependent</u> if there exist scalars $c_1, c_2, \ldots, c_n$ <u>not all zero</u> such that $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$.

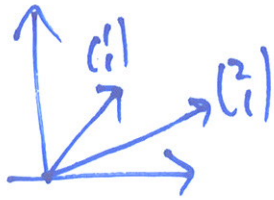
<u>Example 4.26</u> The vectors x_1, x_2, x_3 in the example above are <u>linearly dependent</u> because $3\underset{\sim}{x}_1 + 2\underset{\sim}{x}_2 - \underset{\sim}{x}_3 = \underset{\sim}{0}$.

<u>Terminology 4.27</u> The vectors $v_1, v_2, \ldots, v_n$ in a vector space V are <u>linearly independent</u> if they are <u>not</u> linearly dependent.

<u>That is</u>, <u>if</u> $c_1 v_1 + \cdots + c_n v_n = 0$

<u>then</u> $c_1 = 0, \ldots, c_n = 0$.

<u>Example 4.28</u> The vectors $\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \in \mathbb{R}^2$ are linearly <u>independent</u>.



<u>Why?</u>

If $$c_1\begin{pmatrix}2\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

then $$2c_1 + c_2 = 0$$
$$c_1 + c_2 = 0$$

$$\Rightarrow \quad c_1 = c_2 = 0.$$

<u>Done</u>.

<u>Example 4.29</u> The polynomials p_1 and p_2 defined by

$$p_1(t) = 2+t$$
$$p_2(t) = 1+t$$

are linearly independent.

<u>Why?</u>

Same reason.

As we have seen, linear (in)dependence is closely related to solutions of homogeneous linear systems.

<u>Observation 4.31</u> Let $v_1, \dots, v_n \in \mathbb{R}^n$, and let $A = (v_1 \ v_2 \ \dots \ v_n)$ be the $n \times n$ matrix whose j'th column is v_j.

Then $v_1, \dots, v_n$ are linearly independent if and only if

A is non-singular. (invertible, $\det A \neq 0$)

<u>Reason</u> ~~Proof~~ The equation

$$c_1 v_1 + \dots + c_n v_n$$

is $Ac = 0$, where $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

We know this has a <u>non-trivial</u> solution $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ if and only if A is singular.

That is, $v_1, \dots, v_n$ are <u>linearly dependent</u> if and only if A is <u>singular</u>.

Example 4.32 Determine whether the following vectors are linearly dependent or independent:

$$\begin{pmatrix}-1\\3\\1\end{pmatrix}, \begin{pmatrix}5\\2\\5\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}.$$

Solution linearly dependent

$$\Longleftrightarrow \quad \det\begin{pmatrix}-1 & 5 & 4\\ 3 & 2 & 5\\ 1 & 5 & 6\end{pmatrix} = 0.$$

Now $\det = -1\begin{vmatrix}2 & 5\\ 5 & 6\end{vmatrix} - 5\begin{vmatrix}3 & 5\\ 1 & 6\end{vmatrix} + 4\begin{vmatrix}3 & 2\\ 1 & 5\end{vmatrix}$

$= +13 - 5\times 13 + 4\times 13$

$= 0.$

$\Rightarrow \begin{pmatrix}-1\\3\\1\end{pmatrix}, \begin{pmatrix}5\\2\\5\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}$ are <u>linearly dependent</u>.

Lecture 17

Another way of thinking of <u>linear independence</u>:

<u>Observation 4.33</u> A vector $v \in \mathrm{Span}(v_1, \dots, v_n)$ can be written <u>uniquely</u> as a linear combination of $v_1, \dots, v_n$

if and only if

$v_1, \dots, v_n$ are <u>linearly independent</u>.

<u>If</u>: suppose $v_1, \dots, v_n$ are linearly independent.

If $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$\quad = \beta_1 v_1 + \dots + \beta_n v_n$

then $(\alpha_1 - \beta_1) v_1 + \dots + (\alpha_n - \beta_n) v_n \overset{= v - v}{=} 0$

$\Rightarrow \alpha_i - \beta_i = 0$ for all i.

<u>Only if</u>: suppose $v_1, \dots, v_n$ are linearly dependent, say

$$\gamma_1 v_1 + \gamma_2 v_2 + \dots + \gamma_n v_n = 0$$

with some $\gamma_i \neq 0$.

Then $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

can also be written as

$$v = (\alpha_1 + \gamma_1) v_1 + \dots + (\alpha_n + \gamma_n) v_n$$

and this is a <u>different</u> linear combination because $\alpha_i + \gamma_i \neq \alpha_i$ for the particular i chosen above.

<u>Done.</u>

<u>Remark</u> The case $v = 0$ is just the definition of linearly independent:

$v_1, \dots, v_n$ are linearly independent

if & only if

$\underset{\sim}{0} = 0v_1 + 0v_2 + \dots + 0v_n$ is the <u>only</u> way of writing $\underset{\sim}{0}$ as a linear combination of $v_1, \dots, v_n$.

Section 4.5: Basis and dimension

A basis for V is a minimal spanning set, i.e.

Terminology 4.34 $\{v_1, \dots, v_n\}$ is a basis for V if

- $v_1, \dots, v_n$ are linearly independent
- $\mathrm{Span}(v_1, \dots, v_n) = V$.

Example 4.35 $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

is a basis for $\mathbb{R}^3$

(called the standard basis).

Why? • If $x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

then $x = 0$, $y = 0$, $z = 0$. (Lin. indep.)

• If $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

i.e. $\mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \mathbb{R}^3$.

Example 4.36 $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

is a basis for $\mathbb{R}^3$.

Why? $\det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 1 \neq 0$. **Done**.

In general, a vector space will have many different bases, but (as in this example) every basis has the <u>same number</u> of vectors in it.

This is not obvious.

<u>Helpful fact 4.39</u> Suppose

$$\mathrm{Span}(v_1, \dots, v_n) = V$$

and $u_1, \dots, u_m \in V$, where $m > n$.

Then $u_1, \dots, u_m$ are linearly dependent.

<u>Why?</u> Want to find $c_1, \dots, c_m$ such that

$$c_1 u_1 + \dots + c_m u_m = 0,$$

but at least one $c_i \neq 0$.

This is just a homogeneous system of n linear equations in m unknowns $c_1, \dots, c_m$.

Since $m > n$, there is at least one free variable $\Rightarrow$ there is a non-trivial solution $(c_1, \dots, c_m) \neq (0, 0, \dots, 0)$.

<u>Done</u>.

Consequence 4.40

If V has a basis consisting of n vectors, then **every** basis of V consists of n vectors.

Why? If $\{v_1, \dots, v_n\}$ is a basis, and $\{u_1, \dots, u_m\}$ is a basis,

then $m \not> n$ ($\because$ otherwise $u_1, \dots, u_m$ is linearly dependent).

By symmetry $n \not> m$

$\Rightarrow m = n$.

Terminology 4.41 If V has a basis consisting of n vectors, we say V has **dimension** n, written $\dim V = n$.

(The **zero space** $\{0\}$ has **dimension 0**.)

Example 4.42

$$\dim \mathbb{R}^n = n$$

$$\dim \mathbb{R}^{m\times n} = mn$$

$$\dim P_n = n+1$$

(basis $\{1, t, t^2, \ldots, t^n\}$)

If U is a subspace of V, then $\dim U \le \dim V$

Exercise

Example 4.43 Subspaces of $\mathbb{R}^3$:

- The zero subspace $\{0\}$ of dimension 0.
- 1-dimensional subspaces $\{\alpha v\}$ for any non-zero vector v. (lines through $\underline{0}$)
- 2-dimensional subspaces $\{\alpha v + \beta w\}$ for any two <u>linearly independent</u> vectors v, w. (planes through $\underline{0}$)
- $\mathbb{R}^3$ of dimension 3.

If we know the dimension, then spanning and linear independence are related by:

Observation 4.44 If $\dim V = n$ then

- any set of n linearly independent vectors in V spans V.
- any set of n vectors that span V are linearly independent.

Why? • if $v_1, \ldots, v_n \in V$ are linearly independent and $v \in V$ (we want $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$)

then $v, v_1, v_2, \ldots v_n$ are linearly dependent, say

$$c_0 v + c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0.$$

(not all $c_i = 0$) $\Rightarrow$ $c_0 \neq 0$

$$\Rightarrow v = -\frac{c_1}{c_0} v_1 - \ldots - \frac{c_n}{c_0} v_n$$

$\Rightarrow$ $\operatorname{Span}(v_1, \ldots, v_n) = V$.

• Similarly, if $\text{Span}(v_1, \dots, v_n) = V$ and $v_1, \dots, v_n$ are linearly <u>dependent</u>, say $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$ with $c_i \neq 0$, then $v_i \in \text{Span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$

$$\Rightarrow \text{Span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n) = V$$

Indeed, we can eliminate one vector at a time until we obtain a <u>linearly independent spanning set</u>, with k vectors, where $k < n$.

But then ~~contra~~ $\dim V = k < n$.

<u>Contradiction.</u>

<u>Remark</u> This observation also shows that <u>every spanning set contains a basis</u>.

<u>Remark</u> A basis is not just

- a <u>minimal</u> spanning set,

but also

- a <u>maximal</u> linearly independent set.

<u>Example</u> In $\mathbb{R}^3$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}$$ is a linearly independant set, <u>not</u> spanning

$$\subseteq \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$$ is a maximal linearly independent set, & a minimal spanning set, i.e. a basis.

$$\subseteq \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$$ is a spanning set, but not linearly independent.

Section 4.6 : Coordinates

Example In $\mathbb{R}^n$, with the standard basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

the coordinates of a vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ are just the coefficients in the expansion

$$\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

The same idea gives "coordinates" with respect to any basis.

Terminology 4.47 Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for V. Each $v \in V$ can be written $v = c_1 b_1 + \cdots + c_n b_n$, where $c_1, \ldots, c_n$ are called the coordinates of v ~~with respect to~~ relative to $\mathcal{B}$.

The vector $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$ is the

<u>$\mathcal{B}$-coordinate vector of v</u>,

& is sometimes written $[v]_{\mathcal{B}}$.

<u>Example 4.48</u> $\mathcal{B} = \left\{ \overset{=b_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \overset{=b_2}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \right\}$ for $\mathbb{R}^2$.

If $[x]_{\mathcal{B}} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, what is x ?

<u>Solution</u> $x = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$.

<u>In matrix terms</u> $x = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\underbrace{\begin{pmatrix} -2 \\ 3 \end{pmatrix}}_{[x]_{\mathcal{B}}}$

- call this $P_{\mathcal{B}}$ $\left\{ \begin{array}{l} \text{matrix whose} \\ \underline{\text{columns}} \text{ are the} \\ \text{basis vectors (in order)} \end{array} \right.$

- $P_{\mathcal{B}}$ is <u>invertible</u> because $\mathcal{B}$ is a basis.

<u>The general case</u> (4.51) If $\mathcal{B}$ is a basis for $\mathbb{R}^n$, say $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$, let

$$P_{\mathcal{B}} = \begin{pmatrix} b_1 & b_2 & \ldots & b_n \end{pmatrix}$$

be the $n \times n$ matrix whose columns are $b_1, \ldots, b_n$. Then

- $P_{\mathcal{B}}$ is invertible;
- for every $x \in \mathbb{R}^n$,

$$x = P_{\mathcal{B}} [x]_{\mathcal{B}} . \qquad (*)$$

Why? If $[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ then (by defn.)

$$\begin{aligned} x &= c_1 \underline{b}_1 + \cdots + c_n \underline{b}_n \\ &= \begin{pmatrix} \underline{b}_1 & \ldots & \underline{b}_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\ &= P_{\mathcal{B}} [x]_{\mathcal{B}} . \end{aligned}$$

<u>Done</u>.

<u>Terminology</u> $P_{\mathcal{B}}$ is called the <u>transition matrix</u> from $\mathcal{B}$ to the standard basis.

Consequence 4.52 With the same notation

$$[x]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} x .$$

Example 4.53 Let $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^2$

Find the $\mathcal{B}$-coordinate of $x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Solution $P_{\mathcal{B}} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$$\Rightarrow P_{\mathcal{B}}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow [x]_{\mathcal{B}} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Check $3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} = x.$

Consequence 4.54 If $\mathcal{B}$ and $\mathcal{D}$ are two bases of $\mathbb{R}^n$, then

$$[x]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} P_{\mathcal{D}} [x]_{\mathcal{D}} .$$

Why? $\quad x = P_{\mathcal{D}} [x]_{\mathcal{D}}$

and $[x]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} x$.