Chapter 3: Determinants

Section 3.1 What is a determinant?

Unfortunately this is a difficult question, & the answers I will give you will not be entirely satisfactory.
A full treatment is beyond the scope of this course.

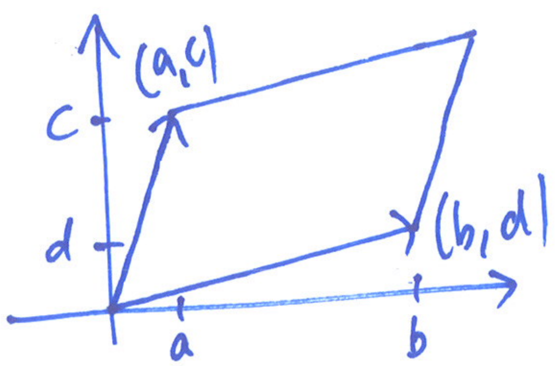
Example If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $\det(A) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

notation

Geometric interpretation of $\det A$

$= \pm$ the area of the parallelogram spanned by vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$



Facts about 2×2 determinants
(which we want to generalise to n×n)

- $\det A \neq 0$ if and only if A is invertible
- $\det(AB) = (\det A).(\det B)$

Example (3×3 determinants)

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$
$$- a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$$
$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= \sum \pm a_{1i}a_{2j}a_{3k}$$ where ijk is one of the 6 orderings of 1,2,3

$+$ for $ijk = 123, 231, 312$

$-$ for $ijk = 132, 213, 321$

General definition of determinants

If $A = (a_{ij})_{n \times n}$ let A_{kl} be the $(n-1)\times(n-1)$ matrix obtained from A by deleting row k and column l.

Then
$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n-1} a_{1n} \det A_{1n}.$$

This gives a <u>recursive</u> definition for $n \times n$ determinants.

<u>Example 3.5</u> Compute $\det A$ where
$$A = \begin{pmatrix} \underline{0} & \underline{0} & 7 & -5 \\ -2 & 9 & 6 & -8 \\ 0 & 0 & -3 & 2 \\ 0 & 3 & -1 & 4 \end{pmatrix}$$

$$\det A = 0 \times \det A_{11} - 0 \times \det A_{12} + 7 \begin{vmatrix} -2 & 9 & -8 \\ 0 & 0 & 2 \\ 0 & 3 & 4 \end{vmatrix} - (-5) \begin{vmatrix} -2 & 9 & 6 \\ 0 & 0 & -3 \\ 0 & 3 & -1 \end{vmatrix}$$

$$= 7\left(-2(-6) - \cancel{9}\right) + 5(-2)9$$

$$= 84 - 90 = -6.$$

Cofactor expansion of determinants

- We defined determinants by expanding across the <u>top row</u>.
- The printed notes from a previous lecturer expand <u>down the first column</u> instead.
- In fact, you can expand <u>across any row</u> or <u>down any column</u>, and you always get the same answer.

(This is unfortunately quite hard to prove rigorously.)

<u>Terminology 3.6</u> If A is a square matrix, the <u>(i,j)-cofactor</u> C_{ij} is the number

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

A_{ij} = matrix obtained by deleting row i & col. j

<u>In particular</u> the cofactor expansion of $\det A$ across the first row is

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}.$$

Fact 3.7 (Cofactor Expansion Theorem)

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$= \sum_{j=1}^{n} a_{ij}C_{ij}$$

for <u>every</u> i

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

$$= \sum_{i=1}^{n} a_{ij}C_{ij}$$

for <u>every</u> j

<u>Reasons</u> We will not prove this here, but most of the important parts of the proof will appear somewhere in this chapter.

Example 3.8 Compute det A by cofactor expansion across the second row, where

$$A = \begin{pmatrix} 4 & -1 & 3 \\ 0 & 0 & 2 \\ 1 & 0 & 7 \end{pmatrix}$$

Solution

$$\det A = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$$

$$= 0 + 0 + 2 \cdot (-1)^{2+3} \begin{vmatrix} 4 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= -2 \cdot (4 \cdot 0 - 1 \cdot (-1))$$

$$= -2.$$

$$C_{23} = (-1)^{2+3} \det \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}.$$

Example 3.9 Compute det A where

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ -2 & 5 & 0 & 0 & 0 \\ 9 & -6 & 4 & -1 & 3 \\ 2 & 4 & 0 & 0 & 2 \\ 8 & 3 & 1 & 0 & 7 \end{pmatrix}$$

Solution Expanding across top row,

$$\det A = 3 . \det \begin{pmatrix} 5 & 0 & 0 & 0 \\ -6 & 4 & -1 & 3 \\ 4 & 0 & 0 & 2 \\ 3 & 1 & 0 & 7 \end{pmatrix}$$

$$= 3 \times 5 \times \det \begin{pmatrix} 4 & -1 & 3 \\ 0 & 0 & 2 \\ 1 & 0 & 7 \end{pmatrix}$$

$$= 3 \times 5 \times (-2) \quad = -30$$

by previous example.

__Fact 3.10__ If A is an upper (or lower) triangular matrix, then $\det A$ is the product of the diagonal entries:

$$\det A = a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{nn}$$

__Reason__ Expand down the first column, & use induction.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & & & \vdots \\ \vdots & & & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

$$\Rightarrow \det A = a_{11} \cdot \det \begin{pmatrix} a_{22} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

$$= a_{11} \cdot a_{22} \cdots a_{nn} \text{ by induction.}$$

Section 3.2 : Properties of determinants

Computing $n \times n$ determinants is exponentially hard.

But we can use elementary row operations to make it easier.

<u>Fact 3.11</u> Let A be a square matrix.

- If B is obtained from A by interchanging two rows, then $\boxed{\det B = -\det A}$
- If B is obtained from A by multiplying one row by α, then $\boxed{\det B = \alpha \det A}$
- If B is obtained from A by adding a multiple of one row onto another then $\boxed{\det B = \det A}$

<u>Some reasons</u> If we multiply first row of A by α to get B,

and $\det A = \sum_{j=1}^{n} a_{1j} C_{1j}$

$$= \sum_{j=1}^{n} a_{1j} (-1)^{1+j} \det A_{1j}$$

then $\det B = \det \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \hline a_{21} & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$

$$= \sum_{j=1}^{n} b_{1j} (-1)^{1+j} \det B_{1j}$$

$$= \sum_{j=1}^{n} (\alpha a_{1j}) (-1)^{1+j} \det A_{1j}$$

$$= \alpha \det (A).$$

<u>Swapping two rows</u> : 2×2 case.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - da = -(ad - bc).$$

General case by induction.

<u>Type $\boxed{B}$ row operations</u> : 2×2 case

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_1 + xR_2} \begin{pmatrix} a + cx & b + dx \\ c & d \end{pmatrix}$$

$$\det \begin{pmatrix} a + cx & b + dx \\ c & d \end{pmatrix} = (a + cx) d - (b + dx) c$$

$= ad - bc.$

Example 3.12

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = - \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & 2 \\ 3 & 12 & 9 \\ 1 & 2 & 1 \end{vmatrix} = 3 \begin{vmatrix} 0 & 1 & 2 \\ 1 & 4 & 3 \\ 1 & 2 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 1 & 0 \\ 7 & 3 & 9 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 \\ 4 & 2 & 9 \\ 0 & -2 & 1 \end{vmatrix}$$

Example 3.13 Compute

$$\begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix} \quad (R_3 + 2R_1)$$

$$= \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 0 & 0 & 0 \\ -5 & -8 & 0 & 9 \end{vmatrix} \qquad (R_3 - R_2)$$

$= 0$ by expanding across 3rd row.

<u>Lectures 11 & 12</u>: no notes currently available.

Lecture 13

A formula for the inverse of a matrix

Let e_j denote the j'th column of I_n.

Then Gauss–Jordan inversion

$$(A|I) \rightsquigarrow (I|A^{-1})$$

specialises to

$$(A|e_j) \rightsquigarrow (I|A^{-1}e_j)$$

which means the solution to $Ax = e_j$ is $x = A^{-1}e_j =$ j'th column of A^{-1}.

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}$ where $A^{-1} = (b_{ij})$

By Cramer's rule, the (i,j)-entry of A^{-1}

= i'th coordinate of x

$$= \frac{\det(A_i(e_j))}{\det A}$$

$$\begin{array}{cc} & \begin{array}{ccc} & i & \end{array} \\ j \to & \begin{pmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ & & \vdots & & \\ & & 1 & & \\ & & \vdots & & \\ a_{n1} & & 0 & & a_{nn} \end{pmatrix} \end{array} = A_i(e_j)$$

Now expanding down the i'th column shows

$$\det(A_i(e_j)) = (-1)^{i+j} \det A_{ji}$$

$$= C_{ji}.$$

$$\Rightarrow \quad A^{-1} = \frac{1}{\det A} \underbrace{\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & & & \vdots \\ \vdots & & & \vdots \\ C_{1n} & & \cdots & C_{nn} \end{pmatrix}}$$

<u>Terminology</u> This matrix is called the <u>adjugate</u> of A, written $\operatorname{adj} A$.

<u>Therefore</u> if A is an invertible matrix,

then $A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$.

<u>Example</u> Compute A^{-1} where

$$A = \begin{pmatrix} 1 & 3 & -1 \\ -2 & -6 & 0 \\ 1 & 4 & -3 \end{pmatrix}.$$

Example Compute the inverse of

$$A = \begin{pmatrix} 1 & 3 & -1 \\ -2 & -6 & 0 \\ 1 & 4 & -3 \end{pmatrix}$$

The $(1,2)$ entry of $A^{-1} = (b_{ij})$ is part of the second column:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \underset{\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}}{\underset{\shortparallel}{A^{-1}}} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

By Cramer's rule

$$x_1 = \frac{\det \begin{pmatrix} 0 & 3 & -1 \\ 1 & -6 & 0 \\ 0 & 4 & -3 \end{pmatrix}}{\det \begin{pmatrix} 1 & 3 & -1 \\ -2 & -6 & 0 \\ 1 & 4 & -3 \end{pmatrix}}$$

$$= \frac{(-1)\det \begin{pmatrix} 3 & -1 \\ 4 & -3 \end{pmatrix}}{\det \begin{pmatrix} 1 & 3 & -1 \\ -2 & -6 & 0 \\ 1 & 4 & -3 \end{pmatrix}} = \frac{(-1)\det A_{21}}{\det A}$$

$$= \frac{(-1) \cdot (-5)}{\det A}$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$\det A = \begin{vmatrix} 1 & 3 & -1 \\ -2 & -6 & 0 \\ 1 & 4 & -3 \end{vmatrix} = 18 - 3.6 - 1(-8+6)$$
$$= 2.$$

$$C_{11} = (-1)^2 \det\begin{pmatrix} -6 & 0 \\ 4 & -3 \end{pmatrix} = 18$$

$$C_{21} = 5.$$

$$C_{31} = (-1)^4 \begin{vmatrix} 3 & -1 \\ -6 & 0 \end{vmatrix} = -6$$

$$C_{12} = -\begin{vmatrix} -2 & 0 \\ 1 & -3 \end{vmatrix} = -6$$

$$C_{22} = +\begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} = -2$$

$$C_{23} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$$

$$C_{32} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$$

$$C_{33} = +\begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{pmatrix} 18 & 5 & -6 \\ -6 & -2 & 2 \\ -2 & -1 & 0 \end{pmatrix}.$$

~~Exercise~~

~~Find the mistake.~~

$$C_{13} = +\begin{vmatrix} -2 & -6 \\ 1 & 4 \end{vmatrix} = -2$$