- In <u>Geometry I</u> you learnt how to reduce <u>all sorts</u> of geometrical problems to <u>one basic problem</u>: solving systems of simultaneous linear equations.

- In <u>Linear Algebra I</u> we will take the same ideas further – many other questions, from physics, engineering, finance, economics, etc. can also be reduced to simultaneous linear equations (often <u>via</u> differential equations)

- In doing so we find that many problems that look different are fundamentally the same. So we build a framework in which they are the same.

- This saves a lot of effort you would otherwise spend learning the "same" thing in different contexts – which is the whole point of mathematics.

- Let us begin by looking again at how to solve simultaneous linear equations — since this is the <u>bedrock</u> on which we are going to build, it's <u>essential</u> that you are completely confident in this material before we move on.

- Exercise classes begin in Week 2
- Information on the course webpage

  ~raw/MTH5112/
- Handing in exercise is compulsory.
- Attendance at exercise classes may be monitored

Exercise classes will be less structured than in the first year - largely you will be left alone to work at your own pace, & ask any questions you wish, about the lecture notes, the exercise, ~~last week's~~ the marking & solutions to last week's exercises, etc. etc. Remember, learning starts with reading your notes, before you attempt the exercises.

- Please try to attend the exercise class for your part of the alphabet. If this is impossible due to timetable clashes you may attend another class instead, providing there is room for you.

In Geometry I you reduced all sorts of problems to ~~solve~~ one problem:

Solving simultaneous linear equations

$$x + y = 2$$
$$2x - 3y = 7$$

etc.

Chapter 1: Systems of linear equations

1.1 Basic terminology / vocabulary

A linear equation in n unknowns is an equation of the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

where $a_1, \dots a_n, b$ are constants (usually real numbers), and $x_1, \dots, x_n$ are "variables" (unknowns).

A system of m linear equations in n unknowns is a collection of m equations:

$$\cancel{a_{1,1}x_1 + a_{2,1}x_2 + \cdots + a_{n,1}x_n = b_1}$$

$$\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m .
\end{aligned}$$

This is an <u>$m \times n$</u> system.

<u>Examples</u>

(a)
$$\begin{aligned} 2x_1 + x_2 &= 4 \\ 3x_1 + 2x_2 &= 7 \end{aligned}$$
2×2 system

(b)
$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 2x_1 - x_2 + x_3 &= 6 \end{aligned}$$
2×3 system

(c)
$$\begin{aligned} x_1 - x_2 &= 0 \\ x_1 + x_2 &= 3 \\ x_2 &= 1. \end{aligned}$$
3×2 system.

<u>A solution</u> to an $m \times n$ system is an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ that satisfies all equations of the system.

(a) has a solution $(1, 2)$.

i.e. $x_1 = 1$, $x_2 = 2$.

(b) has lots of solutions $(3, y, y)$

(c) No solutions.

$x_2 = 1$

So 2nd equation gives $x_1 = 2$

but 1st equation gives $x_1 = 1$.

Contradiction.

A system with no solutions is called <u>inconsistent</u>; a system with at least one solution is <u>consistent</u>

The <u>set</u> of all solutions is the <u>solution set</u>.

~~Pa~~ Two $m \times n$ systems are called <u>equivalent</u> if they have the same solution set.

Example

(a) $5x_1 - x_2 + 2x_3 = -3$

$x_2 = 2$

$3x_3 = 6$

Solution $x_3 = 2$

$x_2 = 2$

Substitute into first equation, to get

$5x_1 - 2 + 4 = -3$

So $5x_1 = -5$

So $x_1 = -1$.

So solution set is $\{(-1, 2, 2)\}$

(b) $$\left.\begin{aligned} 5x_1 - x_2 + 2x_3 &= -3 \\ -5x_1 + 2x_2 - 2x_3 &= 5 \\ 5x_1 - x_2 + 5x_3 &= 3 \end{aligned}\right\} \circledast$$

Solution Add first two equations:

$x_2 = 2$

Subtract first from last:

$0.x_1 + 0.x_2 + 3x_3 = 6$

So $x_3 = 2$, $x_2 = 2$, $x_1 = -1$.

So solution set is $\{(-1, 2, 2)\}$.

Check that it satisfies $\circledast$

So (a) & (b) are equivalent systems.

Useful fact. The following operations on a system of linear equations do not change the solution set:

(i) interchanging two equations.

(ii) multiplying an equation by a non-zero scalar

(iii) adding (a multiple of) one equation to another.

Proof of (iii)

if $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

& $c_1x_1 + c_2x_2 + \dots + c_nx_n = d$

then $ea_1x_1 + ea_2x_2 + \dots + ea_nx_n = eb$

So $(c_1 + ea_1)x_1 + (c_2 + ea_2)x_2 + \dots + (c_n + ea_n)x_n = d + eb$

and conversely:

if $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

~~& $(c_1 + ea_1)x_1 +$~~

& $(c_1 + ea_1)x_1 + (c_2 + ea_2)x_2 + \dots + (c_n + ea_n)x_n = d + eb$

then: $-ea_1x_1+ \cdots = -eb$

So $c_1x_1 + c_2x_2 + \cdots + c_nx_n = d$

Lecture no. 2

Shorthand notation for linear equations (to save time and effort):

Given an $m \times n$ system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n$$

first variable

second variable

the names of the variables are not so important, & we write the coefficient matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

of the system,

and the augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ & & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

Example 1.6

System: $$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 5 \\ 2x_1 \qquad\quad + x_3 &= -1 \end{aligned}$$

augmented matrix: $$\left(\begin{array}{ccc|c} 3 & 2 & -1 & 5 \\ 2 & 0 & 1 & -1 \end{array}\right)$$

Adding the first <u>equation</u> to the second gives
$$(3+2)x_1 + 2x_2 + (-1+1)x_3 = 5-1$$
and corresponds to adding the first <u>row</u> of the matrix to the second row:
$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & 5 \\ 3+2 & 2+0 & -1+1 & 5-1 \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} 3 & 2 & -1 & 5 \\ 5 & 2 & 0 & 4 \end{array}\right)$$

More generally, all the operations we perform on <u>linear equations</u> correspond to operations on <u>rows</u> of the <u>augmented matrix</u>.

We call these

<u>elementary row operations</u>:

<u>Type I</u> interchanging two rows;

<u>Type II</u> multiplying a row by a non-zero scalar;

<u>Type III</u> adding a multiple of one row to another.

Section 1.2: Gaussian elimination

<u>Elimination</u> refers to the elimination of variables, one at a time, by using elementary row operations.

The <u>aim</u> is to get the equations into a "simple" form which is easy to solve.

What does this mean?

In terms of the augmented matrix, we want to put it into a form where each row <u>defines one variable</u> in terms of <u>later</u> variables.

This is called <u>echelon form</u> (or <u>staircase</u> form) – or to be more precise, <u>row echelon form</u>.

The <u>meaning</u> of the term <u>row echelon form</u>.

A matrix is in row echelon form if

(i) all zero rows are at the bottom

(ii) the first non-zero entry in each non-zero row is a 1 (called the <u>leading 1</u> for that row)

<u>and</u> (iii) each leading 1 is to the <u>right</u> of all leading 1s in the rows <u>above</u> it.

<u>Example 1.9</u> The following matrices are in row echelon form

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

— $x_2 + 3x_3 = \cdots$

$x_3 = \cdots$

Second row — 3 × third row

= $(0 \; 1 \; 0)$

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

One can often simplify a matrix further by adding multiples of a row to rows *above* it, to get a matrix in *reduced echelon form*, which means

also (iv) each leading 1 is the *only* entry in its *column*.

Examples

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

A *pivot position* is the position of a leading 1.

A *pivot column* is the corresponding column.

- this corresponds to a *leading variable* in the equations.
- the other variables are *free variables*.

Example 1.10

The augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -4 & 6 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

is in echelon form.

The leading variables are x_1, x_3.

The free variables are x_2, x_4.

In the corresponding system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 - 4x_4 &= 6 \\ x_3 + 2x_4 &= 3 \end{aligned}$$

the free variables are arbitrary,

say $x_4 = \alpha$, $x_2 = \beta$,

& then the leading variables are determined:

$$\begin{aligned} x_3 &= 3 - 2x_4 = 3 - 2\alpha \\ x_1 &= 6 - 2x_2 - 3x_3 + 4x_4 \\ &= 6 - 2\beta - 3(3 - 2\alpha) + 4\alpha \\ &= -3 + 10\alpha - 2\beta. \end{aligned}$$

Important fact Every matrix can be put into row echelon form by a sequence of elementary row operations.

Why? Because there is an algorithm (the Gaussian algorithm) which does it.

How does this algorithm work?

Step 1. If the matrix is zero, stop.

Step 2. Find the first column containing a non-zero entry, & move the first row which has a non-zero entry in that column, to the top.

Step 3. Multiply the top row by a scalar to make this leading entry 1.

Step 4. Add/subtract multiples of this (top) row to rows below it, so that all entries below the leading 1 are 0.

Step 5. Ignore the top row, & carry out Steps 1–4 on the rest of the matrix. Repeat until done.

Lecture no. 3

It is important that this process is done **Systematically**. Otherwise you can end up going round in circles.

Example 1.12 Solve the following system using the Gaussian algorithm:

$$
\begin{aligned}
x_2 + 6x_3 &= 4 \\
3x_1 - 3x_2 + 9x_3 &= -3 \\
2x_1 + 2x_2 + 18x_3 &= 8
\end{aligned}
$$

Solution The augmented matrix is

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 4 \\ 3 & -3 & 9 & -3 \\ 2 & 2 & 18 & 8 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & -3 & 9 & -3 \\ 0 & 1 & 6 & 4 \\ 2 & 2 & 18 & 8 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_1} \left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 2 & 2 & 18 & 8 \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ \hline 0 & 1 & 6 & 4 \\ 0 & 4 & 12 & 10 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & -12 & -6 \end{array}\right) \xrightarrow{R_3 - 4R_2}$$

$$\xrightarrow{-\frac{1}{12}R_3} \left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right).$$

Translating back into equations:

$$\begin{aligned} x_1 - x_2 + 3x_3 &= -1 \\ x_2 + 6x_3 &= 4 \\ x_3 &= \tfrac{1}{2} \end{aligned}$$

Now back-substitution gives

$$x_2 = 4 - 6x_3 = 4 - 3 = 1$$

and $x_1 = -1 + x_2 - 3x_3 = -1 + 1 - \frac{3}{2} = -3/2$

So the solution set is $\left\{ \left(-\frac{3}{2}, 1, \frac{1}{2}\right) \right\}$.

Example 1.12 continued

Echelon form $\left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$

$\xrightarrow{\substack{R_1 - 3R_3 \\ R_2 - 6R_3}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & +1 \\ \hline 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$

$\xrightarrow{R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$

which means

$$x_1 = -\frac{3}{2}$$
$$x_2 = 1$$
$$x_3 = \frac{1}{2}$$

The Gauss–Jordan algorithm takes this one stage further to bring the matrix to reduced echelon form:

Step 1 Use the Gaussian algorithm to bring the matrix to echelon form.

Step 2 Find the row containing the last leading 1, & add suitable multiples of it to the rows above it to make each entry above the leading 1 into 0.

Step 3 Ignore the row just used, and repeat Steps 1–2 on the rest of the matrix.
Repeat until done.

Important fact 1.14

(a) Every matrix can be brought to row echelon form by a sequence of elementary row operations.

(b) Every matrix can be brought to reduced row echelon form by a sequence of elementary row operations.

Reasons (a) is achieved by the Gaussian algorithm

(b) is achieved by the Gauss-Jordan algorithm

Do we need to ~~But~~ be careful about translating this into equations?

For example if the echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

then the third row says $0 = 1$

So the system is <u>inconsistent</u>.

Putting into reduced echelon form

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$ in reduced echelon form.

which still says $0 = 1$ so is inconsistent.

Section 1.3. Special classes of linear systems

Terminology An $m \times n$ ~~$n \times m$~~ linear system is <u>overdetermined</u> if $m > n$; (i.e. no. of equations > no. of variables)
<u>underdetermined</u> if $n > m$.

Fact If an underdetermined system is consistent, then it has <u>infinitely many</u> solutions.

Why? After putting into echelon form, the number of non-zero rows (r say) is $r \leq m$, so $r < n$.

Since r is the number of <u>leading</u> variables, the number of <u>free</u> variables is $n - r \geq n - m > 0$.

Terminology A linear system

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= b_m \end{aligned} \qquad \begin{pmatrix} = 0 \\ = 0 \\ \vdots \\ = 0 \end{pmatrix}.$$

is **homogeneous** if $b_i = 0$ for all i.

and **inhomogeneous** otherwise.

The system obtained from an inhomogeneous system by setting all $b_i = 0$ is the **associated** homogeneous system.

Fact Every homogeneous system is consistent.

Why? Because the zero solution $(0, 0, \dots, 0)$ is always a solution.

Lecture no. 4

- Tutorials start this week:

Weds. 12–1	Geog. 126	surnames	A–K
Thurs. 11–12	BR 3.02	surnames	L–P
Thurs. 2–3	BR 3.02	surnames	Q–Z

- If (and only if) you have a timetable clash you may attend a different session, but note there is no spare capacity in the rooms.

- Exercise Sheets are on the module webpage http://www.maths.qmul.ac.uk/~raw/MTH5112/

- Solutions to be handed in for marking by 12 NOON on FRIDAY, in the RED box in the BASEMENT.

Correction

- It is part of your job in studying mathematics <u>not to believe</u> what the lecturer tells you, but to <u>check it for yourself.</u>
- One person did this and pointed out a mistake after the last lecture. Did anyone else find this mistake? If not, <u>WHY NOT?</u>
- In mathematics, as in life, mistakes are inevitable. What is important is <u>not</u> to avoid mistakes, <u>but</u> to catch them and correct them.
- <u>Proverb</u> "He who never made a mistake never made anything."

Example 1.21

$$3x_1 + 2x_2 + 5x_3 = 2$$
$$2x_1 - x_2 + x_3 = 5$$

is an inhomogeneous system

The associated homogeneous system is

$$3x_1 + 2x_2 + 5x_3 = 0$$
$$2x_1 - x_2 + x_3 = 0.$$

Fact Every homogeneous system has the trivial or zero solution $(0, 0, \ldots, 0)$.

Consequence An underdetermined homogeneous system has non-trivial solutions (& indeed infinitely many solutions)

Reason We have just seen that every homogeneous system is consistent.

We showed in <u>1.19</u> that an underdetermined consistent system has infinitely many solutions. <u>Done</u>.

<u>Special case of $n \times n$ systems.</u>

<u>Fact</u> An $n \times n$ system (is consistent and) has a unique solution
if and only if
the ~~homogeneous~~ associated homogeneous system has <u>only</u> the zero solution (and no non-trivial solutions).

<u>Reason</u> ~~Unique solution~~
There is a unique solution
— if and only if there are n leading variables in the echelon form (& no free variables)
— if and only if the {reduced} row echelon form of the associated homogeneous system is

$$
\begin{array}{l}
x_1 + \cdots = 0 \\
\quad x_2 + \cdots \quad \vdots \\
\qquad x_3 + \cdots \quad \vdots \\
\qquad\quad \ddots \quad = 0 \\
\qquad\qquad x_n = 0
\end{array}
$$

— if and only if the only solution of the homogeneous system is $(0, 0, \ldots, 0)$.