NAME: **FINAL MARK ....../100**
STUDENT NUMBER:

Algebraic Structures I, Test 2010.

Wednesday 24th February 2010 **Duration of Test: 40 minutes.**

This test counts for 10% towards your final examination mark. You may not use calculators. The normal examination regulations apply. Write your answers on the question sheet in the spaces provided. Use the reverse sides for rough work.

Answer all four questions

1. (i) Give the definition of a *ring*.

(ii) Is this a ring: $R = \mathbb{Z}$ with the operations $\oplus$ (addition) and $\otimes$ (multiplication) defined by $a \oplus b = 2(a + b)$ and $a \otimes b = a \times b$? Justify your answer.

(iii) Assuming that, in any ring, the zero is unique, prove that additive inverses (i.e. negatives) are unique.

(iv) Prove that if R is any ring and $a \in R$ then $-(-a) = a$.

2. (i) Give an example of a commutative ring and an example of a non-commutative ring.

(ii) State the *second subring test*.

(iii) Let $S = \{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} | a, b, c \in \mathbb{Z}\}$. Use the subring test to decide if S is a subring of $M_2(\mathbb{R})$.

(iv) State the ideal test.

(v) Let $R = \mathbb{Z}[x]$, the ring of polynomials with integer coefficients, and let S be the subset of polynomials whose constant term is even. Use the ideal test to decide if S is an ideal of R.

3 (i) Give the definition of a *ring homomorphism* $f : R \to S$.

(ii) Define $\ker f$ and show that it is an ideal in R.

(iii) Let $f : \mathbb{Z}[x] \to \mathbb{Z}$ be defined by $f(a_0 + a_1 x + a_2 x^2 + \cdots) = a_0$. Show that f is a ring homomorphism and compute its kernel.

4. In this question, $\mathbb{Z}_n$ denotes the ring of integers modulo n, and $R[X]$ denotes the ring of polynomials with coefficients in R. Also $\mathbb{Z}[i]$ denotes the Gaussian integers $\{a + bi \mid a, b \in \mathbb{Z}\}$.

(i) Define the notion of a *unit* in a commutative ring R with identity, and show that if $a, b \in R$ are units then ab is a unit.

(ii) Define the notion of a *zero divisor* in a commutative ring R. Assuming that R has an identity, show that if a is a unit then a is not a zero divisor.

(iii) List without proof the elements of $\mathbb{Z}_{15}$ which are zero divisors.

(iv) State the definition of an *integral domain*.

(v) Which of the following are integral domains? You need only answer yes or no in each case.

$\mathbb{Z}_7$	$\mathbb{Z}_4[X]$	$\mathbb{Z}[i]$	$M_2(\mathbb{R})$	$\mathbb{Q}$