Queen Mary
University of London

B. Sc. Examination by course unit Sample Exam 2009

MTH5100 Algebraic Structures I

Duration: 2 hours

Date and time: Sample Exam

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): Majid

Question 1 **(a)** Suppose that $\circ$ is an associative binary operation on a set A with identity element e. State what it means for an element $a \in A$ to be *invertible* with respect to $\circ$. State the *cancellation theorem* for $\circ$. [5]

(b) Let $\circ$ be the binary operation on $\mathbb{Z}$ defined by $x \circ y = x(1+y)$ for all $x, y \in Z$. Is this binary operation *associative*? Give a proof of your answer. [2]

(c) Let $\circ$ be the associative binary operation on $M_2(\mathbb{R})$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' & bb' \\ cc' & dd' \end{pmatrix}, \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in M_2(\mathbb{R}).$$

Find an identity element for $\circ$. Which elements of $M_2(\mathbb{R})$ are invertible with respect to $\circ$? [3]

(d) Define a group. Prove using part (a) that each column of the group multiplication table for a finite group contains each element of the group exactly once. [5]

Question 2 **(a)** State a *subring test* and use it to prove that $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a subring of $\mathbb{R}$. [6]

(b) Is S in part (a) a *field*? Justify your answer. You may assume that $(a + b\sqrt{2})(a - b\sqrt{2}) = a^2 - 2b^2$ is not zero for all $a, b \in \mathbb{Q}$ except when $a, b = 0$. [2]

(c) Define what it means for an element of a ring to be a *zero divisor*. Prove that S in parts (a),(b) has no zero divisors. [4]

(d) State what it means for a nonempty subset $I \subseteq R$ of a ring R to be an *ideal*. [2]

(c) Let $\mathcal{P}(U)$ denote the ring of subsets of a set U. State without proof the product and addition in this ring. Show that if $V \subseteq U$ then $I = \mathcal{P}(V)$ is an ideal in $\mathcal{P}(U)$. [4]

Question 3 **(a)** State Lagrange's theorem. [3]

(b) List all the elements of the group $G = GL_2(\mathbb{Z}_2)$. Show that $H = \left\{ e, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \subseteq G$ is a subgroup. State how many distinct right cosets of H there are and list the members of each one. [8]

(c) Use Lagrange's theorem to prove that if a finite group has order a prime number then it is a cyclic group. [6]

Question 4 **(a)** State what is meant by a Euclidean function on an integral domain R. [3]

(b) In any Euclidean domain R, prove that for every ideal I there is an element $x \in R$ such that $I = (x)$. [6]

(c) Let $J = \{m + ni \mid m, n \in \mathbb{Z}\}$ be the ring of Gaussian integers.

(i) List the units in J. Prove that $1 + i \in J$ is irreducible. [3]

(ii) Let the ring homomorphism $\theta : J \to \mathbb{Z}_2$ be defined by $(m + ni)\theta = m + n$ modulo 2 for all $m, n \in \mathbb{Z}$. [Do *not* prove that θ is a homomorphism.] Prove that kernel$(\theta) = (1 + i)$. [4]

Question 5 **(a)** Let I be an ideal in a commutative ring R. State *without proof* the definition of the factor ring R/I. [4]

(b) State the *2nd isomorphism theorem for rings* relating certain subrings and ideals in R to subrings and ideals in R/I. [5]

(c) State what it means for an ideal of a ring to be maximal and state the significance of this for R/I. [3]

(d) Show that $g(X) = X^2 + X + 1$ is irreducible in $\mathbb{Z}_5[X]$ and explain briefly without proofs how this may be used to construct a field of 25 elements. [6]

Question 6 **(a)** State in full the *1st isomorphism theorem* for groups. [6]

(b) The length $l(\alpha)$ of a permutation is defined as the smallest number of transpositions which must be composed to give the permutation. State the length of each of the 6 elements of S_3. [3]

(c) Let $\theta : S_3 \to U(\mathbb{Z})$ be the group homomorphism $\alpha\theta = (-1)^{l(\alpha)}$ for all $\alpha \in S_3$. [Do *not* prove that θ is a homomorphism.] Compute kernel(θ) and image(θ). [4]

(d) Show that S_3 has a subgroup N such that S_3/N is isomorphic to $U(\mathbb{Z})$. [3]

End of Paper