Finite simple groups **Problem sheet 4**

EXERCISE 1. *Show that if x, y and z are three mutually orthogonal purely imaginary quaternions of norm 1, then $xy = \pm z$.*

EXERCISE 2. *Show that in a Moufang loop $(xy)x = x(yx)$. Show also that $(xx)y = x(xy)$ and $y(xx) = (yx)x$. Deduce that any two-generator Moufang loop is a group.*

EXERCISE 3. *Verify that the Moufang identities $(xy)(zx) = (x(yz))x$ and $x(y(xz)) = ((xy)x)z$ and $((xy)z)x = x(y(zx))$ hold in the octonion algebra generated by $i_0, \ldots, i_6$ over any field.*

[Hint: use the reduction to the case $y = i_0$, $z = i_1$, sketched in the notes.]

EXERCISE 4. *Prove that if q is odd then $G_2(q)$ is simple.*

EXERCISE 5. *Show that the maps r and s taking $(x_1, \ldots, x_8)$ to $(-x_1, -x_3, -x_2, x_4, x_5, -x_7, -x_6, -x_8)$ and $(x_2, x_1, -x_6, x_5, x_4, -x_3, x_8, x_7)$, preserve the multiplication table of the split octonion algebra given in the notes.*

EXERCISE 6. *Show that in any associative algebra, the Jordan product $a \circ b$ defined by $a \circ b = \frac{1}{2}(ab+ba)$ satisfies the Jordan identity $((a \circ a) \circ b) \circ a = (a \circ a) \circ (a \circ b)$.*

EXERCISE 7. *Verify that the Albert algebra as defined in the notes is closed under Jordan multiplication. Specifically,*

$$(a, b, c \mid A, B, C) \circ (p, q, r \mid P, Q, R) = (x, y, z \mid X, Y, Z)$$

where $x = ap + \mathrm{Re}\,(C\overline{R} + \overline{B}Q)$ and

$$X = \tfrac{1}{2}(b + c)P + \tfrac{1}{2}(q + r)A + \tfrac{1}{2}(\overline{BR} + \overline{QC}).$$

EXERCISE 8. *Show that if t is a symmetric trilinear form, and $c(x) = t(x, x, x)$, then $24t(x, y, z) = c(x + y + z) + c(x - y - z) + c(-x + y - z) + c(-x - y + z)$.*

Show also that $6t(x, y, z) = c(x + y + z) - c(x + y) - c(y + z) - c(z + x) + c(x) + c(y) + c(z)$.

EXERCISE 9. *Show that the map $(a, b, c \mid A, B, C) \mapsto (a, b, c \mid uA, Bu, \overline{u}C\overline{u})$, where u is an octonion of norm 1, preserves the Jordan product.*