Finite simple groups **Problem sheet 1: October 2008**

Do whichever exercises you feel like doing. Good ones for testing your understanding are 5, 11 and 12.

EXERCISE 1. *For a permutation* $\pi \in S_n$ *define*

$$\varepsilon(\pi) = \prod_{1 \le i < j \le n} \frac{j - i}{j^\pi - i^\pi} \in \mathbb{Q}.$$

Show that $\varepsilon = \pm 1$ *and that* ε *is a group homomorphism from* S_n *onto* $C_2 = \{1, -1\}$. *Hence obtain another proof that the sign of a permutation is well-defined.*

EXERCISE 2. *Let* $G < S_n$ *act transitively on* $\Omega = \{1, \ldots, n\}$ *and let* $H = \{g \in G : a^g = a\}$ *for fixed* $a \in \Omega$. *Prove that* $\phi : a^g \mapsto Hg$ *is a bijection between* Ω *and the set* $G : H$ *of right cosets of* H *in* G.

Prove also that $Hg = \{x \in G : a^x = a^g\}$.

EXERCISE 3. *Prove that the orbits of a group* H *acting on a set* Ω *form a partition of* Ω.

EXERCISE 4. *Show that* A_n *is not* $(n-1)$*-transitive.*

EXERCISE 5. *Let* G *act transitively on* Ω. *Show that the average number of fixed points of the elements of* G *is* 1, *i.e.*

$$\frac{1}{|G|} \sum_{g \in G} |\{x \in \Omega \mid x^g = x\}| = 1.$$

EXERCISE 6. *Verify that the semidirect product* $G :_\phi H$ *is a group, with the given operations. Show that the subset* $\{(g, 1_H) : g \in G\}$ *is a normal subgroup isomorphic to* G, *and that the subset* $\{(1_G, h) : h \in H\}$ *is a subgroup isomorphic to* H.

EXERCISE 7. *Suppose that* G *has a normal subgroup* A *and a subgroup* B *satisfying* $G = AB$ *and* $A \cap B = 1$. *Prove that* $G \cong A{:}_\phi B$, *where* $\phi : B \to \mathrm{Aut} A$ *is defined by* $\phi(b) : a \mapsto b^{-1}ab$.

EXERCISE 8. *Prove that if the permutation* π *on* n *points is the product of* k *disjoint cycles (including trivial cycles), then* π *is an even permutation if and only if* $n - k$ *is an even integer.*

EXERCISE 9. *Determine the number of conjugacy classes in* A_8, *and write down one element from each class.*

EXERCISE 10. *Show that if* $n \ge 5$ *then there is no non-trivial conjugacy class in* A_n *with fewer than* n *elements.*

EXERCISE 11. *Let* S_5 *act on the 10 unordered pairs* $\{a,b\} \subset \{1,2,3,4,5\}$. *Show that this action is primitive. Determine the stabilizer of one of the 10 pairs, and deduce that it is a maximal subgroup of* S_5.

EXERCISE 12. *The previous question defines a primitive embedding of* S_5 *in* S_{10}. *Show that this* S_5 *is not maximal in* S_{10}.

[Hint: construct a primitive action of S_6 *on 10 points, extending this action of* S_5.*]*

EXERCISE 13. *If* $k < \frac{n}{2}$, *show that the action of* S_n *on the* $\binom{n}{k}$ *unordered k-tuples is primitive.*

EXERCISE 14. *If G acts k-transitively on* $\{1,2,\ldots,n\}$ *for some* $k > 1$, *and H is the stabilizer of the point n, show that H acts* $(k-1)$*-transitively on* $\{1,2,\ldots,n-1\}$.

EXERCISE 15. *Let G be the group of permutations of 8 points* $\{\infty,0,1,2,3,4,5,6\}$ *generated by* $(0,1,2,3,4,5,6)$ *and* $(1,2,4)(3,6,5)$ *and* $(\infty,0)(1,6)(2,3)(4,5)$. *Show that G is 2-transitive. Show that the Sylow 7-subgroups of G have order 7, and that their normalisers have order 21. Show that there are just 8 Sylow 7-subgroups, and deduce that G has order 168. Show that G is simple.*

EXERCISE 16. *Let x be an element in* S_n *of cycle type* $(c_1^{n_1},\ldots,c_k^{n_k})$, *where* $c_1,\ldots,c_k$ *are distinct positive integers. Show that the centralizer of x in* S_n *has the shape* $(C_{c_1} \wr S_{n_1}) \times \cdots \times (C_{c_k} \wr S_{n_k})$.

EXERCISE 17. *Show that if* $H \cong \mathrm{AGL}_3(2) \cong 2^3{:}\mathrm{GL}_3(2)$ *is a subgroup of* S_8, *and* $K = H^g$ *where g is an odd permutation, then H and K are not conjugate in* A_8.

EXERCISE 18. *Prove that* $S_k \wr S_2$ *is maximal in* S_{2k} *for all* $k \geq 2$.

EXERCISE 19. *Prove that* $S_k \wr S_m$ *is maximal in* S_{km} *for all* $k,m \geq 2$.

EXERCISE 20. *Prove that the 'diagonal' subgroups of* S_n *(as defined in the notes) are primitive.*

EXERCISE 21. *Show that if H is abelian and transitive on* Ω, *then it is regular on* Ω.

EXERCISE 22. *Use the O'Nan–Scott theorem to write down as many maximal subgroups of* S_5 *as you can. Can you prove your subgroups are maximal?*

EXERCISE 23. *Do the same for* A_5.