**QUEEN MARY,**

**UNIVERSITY OF LONDON**

**M. Sci. Examination 2009 sample**

**MTH734U Topics in Probability and Stochastic Processes: Markov Processes**

**Duration: 3 hours**

**Date and time:**

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*You should attempt all questions. Marks awarded are shown next to the questions.*

*Calculators ARE permitted in this examination. The unauthorized use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.*

*Statistical functions provided by the calculator may be used provided that you state clearly where you have used them.*

*The New Cambridge Statistical Tables are provided.*

*Do not start reading the question paper until instructed to do so.*

*The question paper must not be removed from the examination room.*

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 **TURN OVER**

**Question 1** A simple model for the spread of an epidemic through a large population requires that in a small time interval $(t, t+h]$, each individual has, independently of other individuals, a probability $\lambda h + o(h)$ of infecting exactly one healthy individual, and a probability of $1 - \lambda h + o(h)$ of infecting zero individuals. Independently each individual has a probability $\mu t h + o(h)$ of recovering from the disease, so that the rate of recovery (per individual) is linearly increasing with time.

(a) Obtain the forward equations for $p_n(t) = \Pr[N(t) = n]$, the probability that there are $n$ infected individuals at time $t$. [6]

(b) Show that the mean number of infected individuals $M(t) = \mathrm{E}[N(t)]$ satisfies the ordinary differential equation

$$M'(t) = (\lambda - \mu t)M(t),$$

and solve this equation with initial conditions $N(0) = n_0$ with probability one. [8]

(c) Sketch $M(t)$, finding the maximum and the time at which this maximum is attained, and comment on the uncertainty in this prediction. [6]

**Question 2** A certain type of cell follows a linear birth process with parameter $\lambda > 0$. The partial differential equation for the probability generating function $G(z,t)$ of the number of cells $N(t)$ at time t is given by

$$\frac{\partial G}{\partial t} + \lambda z(1-z)\frac{\partial G}{\partial z} = 0,$$

and it is given that there were $n_0$ cells at $t = 0$.

(a) Show that

$$G(z,t) = z^{n_0}\left(\frac{e^{-\lambda t}}{1 - z + ze^{-\lambda t}}\right)^{n_0}.$$ [8]

(b) State the distribution of $N(t) - n_0$, and hence the mean and variance of $N(t)$. [4]

(c) Establish the identity

$$\Pr[N(t) \geq n] = \Pr[T_n \leq t]$$

connecting $N(t)$ and the cumulative distribution function of $T_n$, the time taken for the population to grow to size $n$. [3]

(d) In the case $n_0 = 2$ and $\lambda = 0.5$, find the probability that the population reaches size 4 at or before time $t = 2$. [5]

**Question 3** (a) State the condition for the existence of an equilibrium distribution in the general birth-death process with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_1, \mu_2, \ldots$, and state the equilibrium distribution when this condition is satisfied. [7]

(b) The queue for cash machines outside a bank has queue discouragement in the sense that the arrival rate is $\lambda_n = \frac{\alpha}{n+1}$ when there are $n$ customers in the queue (including the ones being served). If the service times for machines are independent exponentially distributed random variables with means $\beta^{-1}$, show that

(i) For a single machine, the equilibrium distribution for the queue size is Poisson with mean $\frac{\alpha}{\beta}$. [5]

(ii) For two machines (and a single queue), the equilibrium distribution of queue size is not Poisson and find the mean queue size. [8]

**Question 4** A fast food shop owner has just one server and room for $L$ customers to queue including the one being served. The time taken to serve a customer is exponentially distributed with mean $1/\beta$ and customers arrive according to a Poisson process of rate $k\beta$. It is cold outside, so customers join the queue only if they are able to wait inside the shop, otherwise going elsewhere.

(a) Find the equilibrium distribution of queue length, and hence the proportion of lost customers, and the proportion of time the server is idle. [10]

(b) Improvements to premises now enable twice as many customers to wait in the shop, but the rate of arrivals also doubles. Calculate the proportion of lost customers before and after the improvements in the cases $k = 1/2$, $k = 1$ and $k = 2$ (assume $L = 3$). [10]

 **END OF EXAMINATION**