MTH6136 STATISTICAL THEORY: Exercise Sheet 9

Please try to attempt this question **before** the exercise class. You should post your solution **to parts (a) to (d) only** in the yellow box on the ground floor of the Mathematics Building by **11.00 on Thursday 25 March 2010**. This will be marked and returned to you the following week.

1. Suppose that $Y_1, \ldots, Y_n$ are independent $\mathrm{N}(0, \sigma^2)$ random variables and consider testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 \neq \sigma_0^2$.
(a) Write down the likelihood, $L(\sigma^2; \underline{y})$, and hence find the generalised likelihood ratio, $\Lambda(\underline{y})$.
(b) Show that the critical region of the generalised likelihood ratio test is of the form $R = \{\underline{y} : \sum_{i=1}^n y_i^2 \leq c_1 \text{ or } \sum_{i=1}^n y_i^2 \geq c_2\}$, where c_1 and c_2 are constants chosen to give significance level α, and which obey the extra constraint
$$c_1^{n/2} \exp\left[-\frac{1}{2}c_1/\sigma_0^2\right] = c_2^{n/2} \exp\left[-\frac{1}{2}c_2/\sigma_0^2\right].$$
(c) Given that $\sum_{i=1}^n Y_i^2/\sigma^2 \sim \chi_n^2$ is an exact pivot for σ^2, show that c_1 and c_2 satisfy
$$F_n\left(\frac{c_2}{\sigma_0^2}\right) - F_n\left(\frac{c_1}{\sigma_0^2}\right) = 1 - \alpha$$
where $F_n(.)$ is the distribution function of χ_n^2, in order to give a test with significance level α, and write down the equitailed solution, ignoring the extra constraint in (b).
(d) Hence find c_1 and c_2 in the case $n = 10$, $\sigma_0^2 = 2$ and $\alpha = 0.05$, approximating them to 2 s.f., and show the extra constraint in (b) is not satisfied by this solution.
(e) Show that the power function of this test is given by
$$\beta(\sigma^2) = 1 - \left[F_n\left(\frac{c_2}{\sigma^2}\right) - F_n\left(\frac{c_1}{\sigma^2}\right)\right]$$
and evaluate it using Table 7 to 2 s.f. for $\sigma^2 = \frac{1}{2}, 1, 2, 3, 4$ using linear interpolation if necessary.
(f) What are the main changes to all the above if in fact the underlying population is $N(\mu, \sigma^2)$ with μ unknown?

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1. (a) The likelihood is

$$L(\sigma^2; \underline{y}) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2\right),$$

and so the log-likelihood is

$$\ell(\sigma^2; \underline{y}) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2.$$

Thus, solving the equation

$$\frac{d\ell}{d\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n} y_i^2 = 0,$$

the maximum likelihood estimate of σ^2 is $\hat{\sigma}^2 = \sum_{i=1}^{n} y_i^2/n$. The restricted maximum likelihood estimate of σ^2 under H_0 is $\hat{\sigma}_0^2 = \sigma_0^2$. Hence, the generalised likelihood ratio is

$$\begin{aligned}
\Lambda(\underline{y}) = \frac{L(\hat{\sigma}_0^2; \underline{y})}{L(\hat{\sigma}^2; \underline{y})} &= \frac{(2\pi\hat{\sigma}_0^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\hat{\sigma}_0^2}\sum_{i=1}^{n} y_i^2\right)}{(2\pi\hat{\sigma}^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{n} y_i^2\right)} \\
&= \frac{(\sigma_0^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n} y_i^2\right)}{\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)^{-\frac{n}{2}} \exp\left(-\frac{n}{2}\right)} \\
&= \left(\frac{1}{n\sigma_0^2}\sum_{i=1}^{n} y_i^2\right)^{\frac{n}{2}} \exp\left\{\frac{n}{2}\left(1 - \frac{1}{n\sigma_0^2}\sum_{i=1}^{n} y_i^2\right)\right\}.
\end{aligned}$$

(b) The critical region is $R = \{\underline{y} : \Lambda(\underline{y}) \le a\}$, where a is a constant chosen to give significance level α. By inspection of $\Lambda(\underline{y})$ or by a sketch plot, we see that it will be small when $\sum_{i=1}^{n} y_i^2$ is either small or large. Thus, we reject H_0 if and only if $\sum_{i=1}^{n} y_i^2 \le c_1$ or $\sum_{i=1}^{n} y_i^2 \ge c_2$, where c_1 and c_2 are constants chosen to give significance level α, and which give the <u>same</u> value of the LR statistic, which is the case if the extra constraint holds.

(c) Under H_0, $\sum_{i=1}^{n} Y_i^2/\sigma_0^2 \sim \chi_n^2$. so $\alpha = P\left[\text{reject } H_0 | H_0\right]$

$$= P\left[\sum Y_i^2 \le c_1 \text{ or } \sum Y_i^2 \ge c_2\right] = P\left[\chi_n^2 \le \frac{c_1}{\sigma_0^2}\right] + P\left[\chi_n^2 \ge \frac{c_2}{\sigma_0^2}\right] = F_n\left(\frac{c_1}{\sigma_0^2}\right) + 1 - F_n\left(\frac{c_2}{\sigma_0^2}\right).$$

The equitailed solution is where both probabilities are $\frac{1}{2}\alpha$ so we choose $c_1 = \sigma_0^2\chi^2_{n,1-\frac{\alpha}{2}}$ and $c_2 = \sigma_0^2\chi^2_{n,\frac{\alpha}{2}}$.

(d) The two percentage points are (from Table 8) 3.247 and 20.48 so $c_1 = 6.494 = 6.5$ and $c_2 = 40.96 = 41$ to 2 s.f.
The extra constraint gives LHS=2300, RHS=4100 to 2 s.f so it is clearly not satisfied by the equitailed solution.
(e) We find the power function in a similar way to fixing the size:

$$\beta(\sigma^2) = P\left[\text{reject } H_0|H_1\right] == P\left[\chi_n^2 \le \frac{c_1}{\sigma^2}\right] + P\left[\chi_n^2 \ge \frac{c_2}{\sigma^2}\right] = F_n\left(\frac{c_1}{\sigma^2}\right) + 1 - F_n\left(\frac{c_2}{\sigma^2}\right),$$

so $\beta(\frac{1}{2}) = F_n(13)+1-F_n(82) = 0.78$, $\beta(1) = F_n(6.5)+1-F_n(41) = 0.23$, $\beta(2) = 0.05$ by construction, $\beta(3) = F_n(2.1\dot{6})+1-F_n(13.\dot{6}) = 0.20$, $\beta(4) = F_n(1.625)+1-F_n(10.25) = 0.42$, all to 2 s.f.
(f) The main change is to replace the statistic $\sum Y_i^2$ by $(n-1)S^2$, and to decrease the d.o.f by 1 to $n-1$.