5 Hypothesis Testing

5.1 Definitions

We assume that $Y_1, \ldots, Y_n$ have a joint distribution which depends on the unknown parameters $\theta_1, \ldots, \theta_p$. The set of all possible values of $\underline{\theta}$ is called the **parameter space** Ω.

Example. Suppose that $Y_1, \ldots, Y_n$ are independent $\mathrm{Bin}(m, \pi)$ random variables, where m is known. Then $\underline{\theta} = \pi$ and $\Omega = [0, 1]$.

Example. Suppose that $Y_1, \ldots, Y_n$ are independent Poisson(λ) random variables. Then $\underline{\theta} = \lambda$ and $\Omega = \mathbb{R}^+$.

Example. Suppose that $Y_1, \ldots, Y_n$ are independent $\mathrm{N}(\mu, \sigma^2)$ random variables. Then $\underline{\theta} = (\mu, \sigma^2)^T$ and $\Omega = \mathbb{R} \times \mathbb{R}^+$.

Example. Suppose that $Y_i \sim \mathrm{N}(\beta_0 + \beta_1 x_i, \sigma^2)$ independently for $i = 1, 2, \ldots, n$. Then $\underline{\theta} = (\beta_0, \beta_1, \sigma^2)^T$ and $\Omega = \mathbb{R}^2 \times \mathbb{R}^+$.

A hypothesis restricts $\underline{\theta}$ to lie in ω, where $\omega \subset \Omega$. If we wish to test whether $\underline{\theta} \in \omega$, then we test the **null hypothesis** $H_0 : \underline{\theta} \in \omega$ against the **alternative hypothesis** $H_1 : \underline{\theta} \in \Omega \backslash \omega$.

Example. Suppose that $Y_1, \ldots, Y_n$ are independent Poisson(λ) random variables and that we wish to test $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$. Then $\omega = \{\lambda_0\}$.

Example. Suppose that $Y_1, \ldots, Y_n$ are independent $\mathrm{N}(\mu, \sigma^2)$ random variables and that we wish to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Then $\omega = \{\mu_0\} \times \mathbb{R}^+$.

Example. Suppose that $Y_i \sim \mathrm{N}(\beta_0 + \beta_1 x_i, \sigma^2)$ independently for $i = 1, 2, \ldots, n$ and that we wish to test $H_0 : \beta_1 = 0$ against $H_1 : \beta_1 \neq 0$. Then $\omega = \mathbb{R} \times \{0\} \times \mathbb{R}^+$.

If ω is a single point, the hypothesis is **simple**. Otherwise, the hypothesis is **composite**.

The set of all possible values $\underline{y}$ of the vector of observations $\underline{Y}$ is called the **sample space** S.

Example. Suppose that $Y_1, \ldots, Y_n$ are independent $\mathrm{N}(\mu, \sigma^2)$ random variables. Then $S = \mathbb{R}^n$.

Example. Suppose that $Y_1, \ldots, Y_n$ are independent Poisson(λ) random variables. Then $S = [\mathbf{Z}^+ \cup \{0\}]^n$.

The **critical region** is the subset $R \subseteq S$ such that we reject H_0 if and only if $\underline{y} \in R$.

5.2 Type I and II errors

There are two errors in hypothesis testing. A **type I error** is to reject H_0 when H_0 is true and a **type II error** is to accept H_0 when H_0 is false.

When H_0 is simple, that is, $H_0 : \underline{\theta} = \underline{\theta}_0$, the **significance level** or **size** of the test is defined to be

$$\begin{aligned} \alpha &= P(\text{type I error}) = P(\underline{Y} \in R | H_0 \text{ true}) \\ &= P(\underline{Y} \in R | \underline{\theta} = \underline{\theta}_0). \end{aligned}$$

When H_0 is composite, that is, we have $H_0 : \underline{\theta} \in \omega$, then we have a significance function $\alpha(\underline{\theta}) = P(\underline{Y} \in R | \underline{\theta})$ which depends on the particular value in ω taken by $\underline{\theta}$. In this case, the **size** of the test or **significance level** is defined to be $\alpha = \max_{\underline{\theta} \in \omega} \alpha(\underline{\theta})$.

When H_1 is simple, that is, $H_1 : \underline{\theta} = \underline{\theta}_1$, the **power** of the test is defined to be

$$\begin{aligned} \beta &= 1 - P(\text{type II error}) = P(\underline{Y} \in R | H_1 \text{ true}) \\ &= P(\underline{Y} \in R | \underline{\theta} = \underline{\theta}_1). \end{aligned}$$

When H_1 is composite, that is, we have $H_1 : \underline{\theta} \in \Omega \backslash \omega$, then we have a **power function** $\beta(\underline{\theta}) = P(\underline{Y} \in R | \underline{\theta})$ which depends on the value in $\Omega \backslash \omega$ taken by $\underline{\theta}$.