MTH6136 STATISTICAL THEORY
Mid-Term Revision

Name: .... **Number:** ....

Please answer **all** the questions and write your answers on the question sheet. The marks for questions are indicated. The total mark is 100 and the duration is 50 minutes.

1. Suppose that the random variable Y has mean θ and probability density function

$$f_Y(y) = \frac{1}{\sqrt{2\pi y^3}} \exp\left\{-\frac{(y-\theta)^2}{2y\theta^2}\right\}, \quad y > 0,$$

   where $\theta > 0$ is an unknown parameter. Show that this distribution is a member of the exponential family. [15 marks]

2. $Y_1, \ldots, Y_n$ are independent random variables, each with this distribution, i.e. a *random sample*. Write down a complete sufficient statistic for θ. [5 marks]

3. Show that the maximum likelihood estimator of θ is $\hat{\theta} = \overline{Y}$. Is this also the method of moments estimator (MOME)? [15 marks]

4. Show that $\hat{\theta}$ is an unbiased estimator of θ. [10 marks]

5. Show that the Cramér-Rao lower bound for unbiased estimators of θ is given by $\text{CRLB}(\theta) = \theta^3/n$. [20 marks]

6. Now consider estimating the parameter $1/\theta$. Given that $E(1/\hat{\theta}) = 1/\theta + 1/n$ and $\text{var}(1/\hat{\theta}) = 1/(n\theta) + 2/n^2$, show that $1/\hat{\theta}$ is a consistent estimator of $1/\theta$. [10 marks]

7. Find the efficiency for sample size n of $1/\hat{\theta}$, and hence show it is an efficient estimator of $1/\theta$. [15 marks]

8. Find the minimum variance unbiased estimator of $1/\theta$. [10 marks]

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Solutions to Mid-Term Revision

1. The probability density function of Y may be written as

$$\begin{aligned} f_Y(y) &= \exp\left\{-\frac{1}{2}\log(2\pi y^3) - \frac{1}{2y\theta^2}(y^2 - 2y\theta + \theta^2)\right\} \\ &= \exp\left\{-\frac{1}{2\theta^2}y + \frac{1}{\theta} - \frac{1}{2y} - \frac{1}{2}\log(2\pi y^3)\right\}. \end{aligned}$$

Hence, the distribution is a member of the exponential family with $a(\theta) = -1/(2\theta^2)$, $b(y) = y$, $c(\theta) = 1/\theta$ and $d(y) = -1/(2y) - \log(2\pi y^3)/2$.

2. A complete sufficient statistic for θ is $\sum_{i=1}^n b(Y_i) = \sum_{i=1}^n Y_i$.

3. The likelihood is

$$L(\theta; \underline{y}) = \left(\prod_{i=1}^n \frac{1}{\sqrt{2\pi y_i^3}}\right) \exp\left\{-\frac{1}{2\theta^2}\sum_{i=1}^n \frac{1}{y_i}(y_i - \theta)^2\right\},$$

and so the log-likelihood is

$$\ell(\theta; \underline{y}) = -\frac{1}{2}\sum_{i=1}^n \log(2\pi y_i^3) - \frac{1}{2}\sum_{i=1}^n \left(\frac{y_i}{\theta^2} - \frac{2}{\theta} + \frac{1}{y_i}\right).$$

Thus, we have

$$\frac{d\ell}{d\theta} = \sum_{i=1}^n \left(\frac{y_i}{\theta^3} - \frac{1}{\theta^2}\right) = 0 \Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^n y_i,$$

so that the maximum likelihood estimator of θ is $\hat{\theta} = \sum_{i=1}^n Y_i/n = \overline{Y}$. Since this equates the first sample and population moments (about zero), $\hat{\theta}$ is also the MOME.

4. Since we may write

$$E(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^n E(Y_i) = \frac{1}{n}n\theta = \theta,$$

$\hat{\theta}$ is an unbiased estimator of θ.

5. First note that, since $g(\theta) = \theta$, we have $dg/d\theta = 1$. Next, we know that

$$\frac{d\ell}{d\theta} = \frac{1}{\theta^3}\sum_{i=1}^n y_i - \frac{n}{\theta^2}$$

and

$$\frac{d^2\ell}{d\theta^2} = -\frac{3}{\theta^4}\sum_{i=1}^n y_i + \frac{2n}{\theta^3}.$$

It follows that

$$E\left(-\frac{d^2\ell}{d\theta^2}\right) = \frac{3}{\theta^4}\sum_{i=1}^n E(Y_i) - \frac{2n}{\theta^3} = \frac{3}{\theta^4}n\theta - \frac{2n}{\theta^3} = \frac{n}{\theta^3},$$

and hence

$$\text{CRLB}(\theta) = \frac{1}{E\left(-\frac{d^2\ell}{d\theta^2}\right)} = \frac{\theta^3}{n}.$$

6. Since $E(1/\hat{\theta}) \to 1/\theta$ and $\text{var}(1/\hat{\theta}) \to 0$ as $n \to \infty$, $1/\hat{\theta}$ is a consistent estimator of $1/\theta$.

7. Let $g(\theta) = 1/\theta$, so that $dg/d\theta = -1/\theta^2$. Then the Cramér-Rao lower bound for unbiased estimators of $1/\theta$ is

$$\text{CRLB}\left(\frac{1}{\theta}\right) = \frac{(-1/\theta^2)^2}{n/\theta^3} = \frac{1}{n\theta}.$$

   The efficiency of $1/\hat{\theta}$ is

$$\frac{\text{CRLB}(1/\theta)}{\text{var}(1/\hat{\theta})} = \frac{1/(n\theta)}{1/(n\theta) + 2/n^2} = (1 + 2\theta/n)^{-1}.$$

   This has limit 1 as $n \to \infty$, so $\hat{\theta}$ is efficient.

8. Since $E(1/\hat{\theta} - 1/n) = 1/\theta$ and $1/\hat{\theta} - 1/n$ is a function of the complete sufficient statistic, it is the minimum variance unbiased estimator of $1/\theta$, by the Lehmann-Scheffé theorem.