Normal example   MTH6136p13a.pdf   ~~IV~~

Now as $\hat{\phi} = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$,

$$\mathrm{Var}[\hat{\phi}] = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}[Y_i^2]$$

$$= \frac{1}{n}\mathrm{Var}[Y^2]$$

~~since~~ (as Y_i^2 $i=1,\dots,n$ are ~~independent~~ ~~and~~ IID (independent and identically distributed $N(\mu,\sigma^2)$))

and $\mathrm{Var}[Y^2] = E[Y^4] - (E[Y^2])^2$ Now $E[Y^2] = \mu^2 + \sigma^2$

+ We know for the Normal that

$$E[(Y-\mu)^3] = 0$$
$$E[(Y-\mu)^4] = 3\sigma^4 \quad \text{so}$$

(Prob. 2 or differentiate mgf of $Y-\mu$ ie $e^{\frac{1}{2}\sigma^2 t^2}$ 3 and 4 times w.r.t. t & put $t=0$)

$$(Y-\mu)^3 = Y^3 - 3Y^2\mu + 3Y\mu^2 - \mu^3$$
$$\Rightarrow \quad 0 = E[Y^3] - 3\mu(\mu^2+\sigma^2) + 3\mu^3 - \mu^3$$
$$\longrightarrow \quad \underline{E[Y^3] = 3\mu\sigma^2 + \mu^3}$$

and

$$(Y-\mu)^4 = (Y^2 - 2Y\mu + \mu^2)^2$$
$$= (Y^4 - 4Y^3\mu + 6Y^2\mu^2 - 4Y\mu^3 + \mu^4)$$
$$\longrightarrow \quad 3\sigma^4 = E[Y^4] - 4(3\mu\sigma^2 + \mu^3)\mu + 6\mu^2(\mu^2+\sigma^2) - 4\mu^4 + \mu^4$$
$$\longrightarrow \quad E[Y^4] = \underline{3\sigma^4 + 6\mu^2\sigma^2 + \mu^4}$$

Hence

$$\mathrm{Var}[\hat{\phi}] = \frac{1}{n}\left[3\sigma^4 + 6\mu^2\sigma^2 + \mu^4 - (\mu^2+\sigma^2)^2\right]$$
$$= \frac{1}{n}\left[2\sigma^4 + 4\mu^2\sigma^2\right] = \frac{2\sigma^2}{n}(2\mu^2+\sigma^2)$$

which is the CRLB

Hence $\hat{\phi}$ is the MVUE

Is there an easier way to prove this?

Yes(!) by finding sufficient and complete statistics. (See later)

or using the condition of attaining the CRLB: see lectures