6 Simulation

6.1 Introduction

What is simulation? We are all familiar with the idea of a flight simulator. It enables trainee pilots to learn how to pilot a plane without the risk of crashing. It also enables experienced pilots to practice landing at unfamiliar airports or to experience emergency scenarios with no risk.

A statistical simulation can be similar. We can model a real-life situation mathematically and then change things to see the effect.

Example Suppose a bank has a number of clerks serving customers. Customers arrive at the bank and have to wait for a clerk to be free to serve them. There are a number of possible queueing disciplines.

1. One common queue, when a clerk is free the customer at the head of the queue is served.

2. There is one queue for each clerk, each customer on arrival chooses the shortest queue and remains in it.

3. As for 2 but each customer chooses a queue at random.

4. As for 2 but customers are allocated to a queue in rotation.

5. Variants of 2, 3 and 4 in which queue changing is allowed if a queue becomes empty.

(Note: some of these queue disciplines might be unrealistic in a bank, but might apply to components in a factory where the clerk represents part of the manufacturing process.)

We can model these situations mathematically. We need to know about how customers arrive and how long it takes to serve them, both are usually random quantities. We are interested in the time the customers have to wait in the bank and what proportion of time the clerks are idle. Some simple queues can be analysed completely but many need to be simulated. Using our mathematical model we can look at the effect of changing the queueing discipline or the number of servers. As with the aircraft simulator we can do this without risking a disaster. Only if we are confident that a proposed change will not have a bad effect would we put it into operation.

To carry out the simulation we have to be able to generate realisations of a given random variable. In this chapter we shall concentrate on how to do this.

Another area in which simulation is useful is to illustrate some statistical ideas and theorems. We shall see some examples later in the course. MINITAB provides for simulation from a wide range of distributions and we shall see how this works in future practicals.

6.2 Generating continuous uniform random variables

There are a number of algorithms for generating values from a continuous uniform distribution on the unit interval. I shall write this distribution as $U(0,1)$. Many are based on congruential generators of the form

$$x_{n+1} = a + bx_n \qquad \bmod m,$$

$$u_n = \frac{x_n}{m}$$

Much effort goes into finding choices of a,b and m to ensure the generator has good statistical properties. We will not go further into this here.

For our purposes we may make use of the random digits given in Table 27 of the New Cambridge Statistical Tables. Given a sequence of digits $d_1, d_2, \ldots$, it is possible to show that

$$u = \sum_i d_i \times 10^{-i}$$

has a $U(0,1)$ distribution. In practice we can truncate after say 4 decimal places to provide suitable values.

Example If we use the first four columns in Table 27 we get values 0.8442, 0.2887, 0.6412, 0.4941, 0.0646 and so on. We may use these values if we require a sample from $U(0,1)$.

It is also possible to devise schemes for generating random digits using dice or coins. For example suppose we toss a fair coin four times and record the sequence of heads and tails. If we associate a head with 0 and a tail with 1 and consider the resulting four figure number as a binary number we will have numbers 0 (0000) to 15 (1111). If we reject any number greater than 9 then this will provide a method of generating random digits.

6.3 Functions of random variables

6.3.1 One dimensional change of variable

We begin by using a result you saw in Probability I

Suppose Y is a random variable with density $f_Y(y)$. If $X = h(Y)$ where h is a one to one transformation with inverse h^{-1} then the density for X is

$$f_X(x) = f_Y(h^{-1}(x)) \mid \frac{dy}{dx} \mid .$$

Example As a simple example suppose the density of Y is

$$f_Y(y) = 2y \;\; 0 < y < 1$$

and zero otherwise. Let $X = 4 - 3Y$. The inverse transformation is $y = \frac{4-x}{3}$ and $\frac{dy}{dx} = -\frac{1}{3}$ thus

$$f_X(x) = 2\left(\frac{4-x}{3}\right) \times \frac{1}{3}.$$

We also need to transform the range, when $y = 0, x = 4$ and when $y = 1, x = 1$ so we have

$$f_X(x) = \frac{2}{9}(4-x) \;\; 1 < x < 4.$$

Example A more useful, although again simple example, is the following.

Let U be uniformly distributed on the interval $[0,1]$, written $U(0,1)$, with density equal to one on the unit interval. Let $X = 1 - U$, what is the distribution of X?

The inverse transformation is $u = 1 - x$ and $\frac{du}{dx} = -1$ thus

$$f_X(x) = 1 \times 1 = 1,$$

and we see that the unit interval is mapped to the unit interval by this transformation. Thus X also has a $U(0,1)$ distribution.

Example Let U be uniformly distributed on the interval $[0,1]$. Let $X = -\frac{1}{\lambda}\log_e U$, what is the distribution of X?

The inverse transformation is $u = \exp(-\lambda x)$ and $\frac{du}{dx} = -\lambda \exp(-\lambda x)$ thus

$$f_X(x) = 1 \times \lambda \exp(-\lambda x).$$

The unit interval is mapped to $(0,\infty)$ and we see that X has an exponential distribution $Exp(\lambda)$. Thus it is straightforward to generate from an exponential distribution given values from a uniform distribution on the unit interval as supplied by one of the congruential generators or the Tables.

6.4 Generating arbitrary random numbers

6.4.1 Simulating discrete random variables

Discrete Uniform

Let U be a continuous random variable, uniformly distributed on $(0,1)$. Let $V = \lfloor kU \rfloor$ where $\lfloor x \rfloor$ is the floor of x, that is the integer below x. Then, since $\Pr\{U = 1\} = 0$, we have $0 \le V \le k-1$ and

$$\begin{aligned}
\Pr\{V = v\} &= \Pr\{v \le kU < v+1\} \\
&= \Pr\left\{\frac{v}{k} \le U < \frac{v+1}{k}\right\} \\
&= \begin{cases} \frac{1}{k} & ; \quad v = 0,1,\ldots,k-1 \\ 0 & ; \quad \text{otherwise.} \end{cases}
\end{aligned}$$

Thus, to simulate a random variable from the discrete uniform distribution on $\{0, 1, \ldots, k-1\}$, we could simulate a continuous uniform U and then find $\lfloor kU \rfloor$. To simulate a random variable from the discrete uniform distribution on $\{1, \ldots, k\}$, we would find $\lfloor kU \rfloor + 1$.

General discrete distribution

Suppose we wish to simulate a discrete rv with probability function

$$f(x) = \begin{cases} p_n & ; \quad x = x_n,\ n = 1, \ldots, k \\ 0 & ; \quad \text{otherwise,} \end{cases}$$

where $x_1 < \cdots < x_k$. Let $F(x)$ be the distribution function, which is given by

$$F(x) = \begin{cases} 0 & ; \quad x < x_1 \\ \sum_{j=1}^{n} p_j & ; \quad x_n \leq x < x_{n+1},\ 1 \leq n \leq k-1 \\ 1 & ; \quad x \geq x_k. \end{cases}$$

Let U be continuous and uniformly distributed on $(0, 1)$. Let

$$X = \begin{cases} x_1 & ; \quad 0 \leq U < F(x_1) \\ x_n & ; \quad F(x_{n-1}) \leq U < F(x_n),\ 2 \leq n \leq k. \end{cases} .$$

Then

$$\begin{aligned} \Pr\{X = x\} &= \begin{cases} F(x_1) & ; \quad x = x_1 \\ F(x_n) - F(x_{n-1}) & ; \quad x = x_n,\ 2 \leq n \leq k. \end{cases} \\ &= f(x). \end{aligned}$$

This method is sometimes called the table look-up method.

Example Bernoulli distribution.

For a Bernoulli distribution with $P(X = 0) = 1 - p$ and $P(X = 1) = p$, then if u is a random value from $U(0, 1)$ we take $x = 0$ if $0 < u \leq 1 - p$ and $x = 1$ if $1 - p < u \leq 1$.

To find two values from a Bernoulli distribution with $p = 0.4$ the first two values from Tables 27 are 0.8442 and 0.2887. So the two values are 1 and 0.

Example Explain how to generate observations from a binomial distribution with $n = 5$ and $p = 0.5$, if you have a supply of observations from a $U(0, 1)$ distribution. Illustrate your answer by generating 6 such values using the numbers in Table 27.

Using Table 1 in New Cambridge Statistical Tables we see that the cumulative binomial probabilities are $P(X \leq 0) = 0.0313$, $P(X \leq 1) = 0.1875$, $P(X \leq 2) = 0.5000$, $P(X \leq 3) = 0.8125$, $P(X \leq 4) = 0.9688$, $P(X \leq 5) = 1.00000$.

Thus if we generate a u from $U(0, 1)$ we obtain an x as follows

$$\begin{array}{ll} 0.0000 \leq u < 0.0313 & x = 0 \\ 0.0313 \leq u < 0.1875 & x = 1 \\ 0.1875 \leq u < 0.5000 & x = 2 \\ 0.5000 \leq u < 0.8125 & x = 3 \\ 0.8125 \leq u < 0.9688 & x = 4 \\ 0.9688 \leq u < 1.0000 & x = 5 \end{array}$$

Using the first 4 columns of Table 27 of we obtain the u values 0.8442, 0.2887, 0.6412, 0.4941, 0.0646 and 0.7556. and hence the x values 4, 2, 3, 2, 1, 3.

(Note that because of the symmetry of the binomial with $p = 0.5$, we could just as well reverse the order of the x values in our table and this would have resulted in x values 1, 3, 2, 3, 4, 2.)

For special distributions, there are usually clever tricks available. These will be needed especially when the discrete random variable takes on an infinite number of values. There are also techniques which generate general discrete random numbers efficiently – these use "look-up tables" which have been generated in special ways. For example, if we arrange the values of the p_j in descending order, we are more likely to get a match early on and will have to do fewer comparisons.

Example Geometric distribution Although it is possible to use the table look up method for a geometric distribution we can utilise its definition as the number of trials till the first success occurs in an independent sequence of Bernoulli trials, each with probability p of success. We can use the method given above to generate values from a Bernoulli distribution. Then count how many you need to generate to get to the first 1. This is the value x_1. Repeat to obtain as many as are needed.

So to generate values from a geometric disribution with $p = 0.4$ we find a number of U(0,1) values, e.g. 0.8442, 0.2887, 0.6412, 0.4941, 0.0646, .7556, .0935, .7381, .4969, .6460, .9305.

The corresponding values from the Bernoulli distribution are

1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1

So the random values from the geometric are

1, 2, 3, 2, 2, 1.

6.4.2 Simulating continuous random variables

Suppose we wish to simulate a random variable which has distribution function $F(x)$, where $F(\alpha) = 0$, $F(\beta) = 1$,. $F(x)$ is strictly increasing on $[\alpha, \beta]$, α is possibly $-\infty$ and β is possibly ∞. Let U be a continuous random variable which is distributed uniformly on $(0,1)$, and let F^{-1} be the inverse of F, i.e., $F^{-1} : [0,1] \to [\alpha, \beta]$ and $F^{-1}(y) = x$, if $y = F(x)$ and $y \in (0,1)$. Let $X = F^{-1}(U)$. Then the distribution function F_X of X is given by

$$
\begin{aligned}
F_X(x) &= \Pr\{X \le x\} \\
&= \Pr\left\{F^{-1}(U) \le x\right\} \\
&= \Pr\{U \le F(x)\} \\
&= \begin{cases} 0 & ; \quad x \le \alpha \\ F(x) & ; \quad \alpha < x \le \beta \\ 1 & ; \quad x > \beta \end{cases} \\
&= F(x).
\end{aligned}
$$

Thus, if there is an expression for F^{-1}, we may simulate a random variable with distribution function F by simulating a uniformly distributed random variable U and calculating $F^{-1}(U)$.

Example Suppose we wish to simulate an exponential random variable with mean λ^{-1}. The distribution function is

$$F(x) = \begin{cases} 1 - e^{-x\lambda} & ; \quad x > 0 \\ 0 & ; \quad x \leq 0. \end{cases}$$

We thus let $\alpha = 0$ and $\beta = \infty$. The inverse function is given by

$$F^{-1}(y) = -\frac{1}{\lambda} \log(1 - y).$$

Hence $X = -\lambda^{-1} \log(1 - U)$ has the desired distribution.

Note that we have shown that $1 - U$ is also uniform on $[0, 1]$, so this result agrees with the previous result.

Example Standard normal distribution

Let U be a random value from $U(0,1)$, then we want to find z such that $z = \Phi^{-1}(u)$, where Φ is the cdf of the standard normal distribution ($N(0,1)$). hence we want z such that $\Phi(z) = u$. The cdf of the normal does not have a closed form but we can utlise the tables of it.

If $u \geq 0.5$, look up u in the interior of Table 4 so that $\Phi(z) = u$. Read off the value of z. this is a random value from an $N(0,1)$ distribution.

If $u < 0.5$ then the value z is negative and is not given in the Table. We use the symmetry of the normal. Find w such that $\Phi(w) = 1 - u$ in the same way as above. Then take $z = -w$.

Taking the first 2 values in Table 27 we have $u_1 = 0.8442$ and $u_2 = .2887$.

Now $\Phi(1.01) = .8438$ and $\Phi(1.02) = .8461$.

$$\frac{.8442 - .8438}{.8461 - .8438} = \frac{4}{23} = 0.17$$

so our $z_1 = 1.01 + .0017 = 1.0117$.

Similarly $1 - u_2 = .7113$. Now $\Phi(.55) = ..7088$ and $\Phi(.56) = .7123$.

$$\frac{.7113 - .7088}{.7123 - .7088} = \frac{25}{35} = 0.71$$

so our $z_2 = -(0.55 + 0.0071) = -0.5571$.

Example $N(\mu, \sigma^2)$ distribution

We can use the result that if $Z \sim N(0,1)$ then $\sigma Z + \mu \sim N(\mu, \sigma^2)$. So to simulate two values from $N(1,4)$ we can take our values z_1 and z_2 multiply by 2 and add 1. So $x_1 = 2 \times 1.0117 + 1 = 3.0234$ and $x_2 = 2 \times -0.5571 + 1 = -0.1142$.

Note that although we can use these methods to find random normal values they are not suitable if we want to generate many such values. there are far better algorithms to do so using methods beyond the scope of this course. Moreover Table 28 actually gives us random standard normal values calculated using one of these algorithms.

6.5 Using simulated values

We shall use simulation later in the course to illustrate some statistical ideas we meet. For now I shall give a very simple example which shows how simulation can be used in practical problems.

Some queues can be studied theoretically and even quite complicated scenarios can yield some conclusions. But even in these cases simulation can provide a method of investigating the behaviour of the queue and the effect of changes.

Consider the following two server queue. The distribution of service time is exponential with mean 1.5. Arrivals are scheduled regularly at times $1, 2, 3, \ldots,$ but actually arrive at the scheduled time plus an error where the error is normally distributed with mean 0 and standard deviation 0.5. The following table illustrates how the queue behaves for the first 12 customers. Note that all quantities are given to one decimal place which is adequate for this situation. It is assumed that, if both servers are free, the customer is served by server number 1.

Customer	Arrival error	Arrival time	Service time	Enter S1	Depart S1	Admit S2	Depart S2	Total time
1	$+0.4$	1.4	0.2	1.4	1.6			0.2
2	-0.3	1.7	2.7	1.7	4.4			2.7
3	-0.2	2.8	1.7			2.8	4.5	1.7
4	$+0.4$	4.4	4.3	4.4	8.7			4.3
5	$+0.4$	5.4	1.7			5.4	7.1	1.7
6	-0.3	5.7	0.6			7.1	7.7	2.0
7	-0.3	6.7	2.7			7.7	10.4	3.7
8	-0.4	7.6	1.0	8.7	9.7			2.1
9	-0.5	8.5	0.6	9.7	10.3			1.8
10	$+0.6$	10.6	3.6	10.6	14.2			3.6
11	-0.1	10.9	0.7			10.9	11.6	0.7
12	-0.1	11.9	0.9			11.9	12.8	0.9

Note that the effect of changes can easily be investigated. For example suppose that customers are allocated to the servers alternately rather than to the first one that is free. Then the following table shows what happens to the first 12 customers assuming the same Arrival errors and service times.

Customer	Arrival error	Arrival time	Service time	Enter S1	Depart S1	Admit S2	Depart S2	Total time
1	$+0.4$	1.4	0.2	1.4	1.6			0.2
2	-0.3	1.7	2.7			1.7	4.4	2.7
3	-0.2	2.8	1.7	2.8	4.5			1.7
4	$+0.4$	4.4	4.3			4.4	8.7	4.3
5	$+0.4$	5.4	1.7	5.4	7.1			1.7
6	-0.3	5.7	0.6			8.7	9.3	3.6
7	-0.3	6.7	2.7	7.1	9.8			3.1
8	-0.4	7.6	1.0			9.3	10.3	2.7
9	-0.5	8.5	0.6	9.8	10.4			1.9
10	$+0.6$	10.6	3.6			10.6	14.2	3.6
11	-0.1	10.9	0.7	10.9	11.6			0.7
12	-0.1	11.9	0.9			14.2	15.1	3.2

Note that the average time spent queueing (including service time) increases from 2.12 to 2.45.