MTH5119 2009 SOLUTIONS
(also resit/retake: MAS234 2009 SOLUTIONS)

Similar example in lectures & test

Q1 (a) Initial sample: $\{1,2,3\}$

$k=3$ Choose ('new' unit) 4 (with) prob. $n/_{k+1} = 3/4 = 0.75$
$u = 0.75\ldots\ldots > 3/4$ so do not select (unit 4)

$k=4$ Choose (new unit) 5 with prob $n/_{k+1} = 3/5 = 0.6$
$u = 0.6\ldots\ldots > 0.6$ so do not select (unit 5)

$k=5$ Choose (new unit) 6 (with) prob. $n/_{k+1} = 3/6 = 0.5$
$u = 0.2\ldots < 0.5$ so select unit 6

replace: intervals 0 – ① – $1/3$ – ② – $2/3$ – ③ – 1 ? $u = 0.60\ldots\ldots < 2/3,\ > 1/3$ so replace unit ②

sample is now $\{1,3,6\}$

$k=6$ Choose new unit 7 with prob $n/_{k+1} = 3/7$ $u = .1\ldots$
so select unit 7 and replace: intervals ① – $1/3$ – ③ – ⑥
$u = .35$, replace unit ③.
Final sample is $\{1,6,7\}$ 11

(b) sample values are $2.5, 3.0, 2.3$ so $\hat{T} = N\bar{y} = \frac{7}{3}\times 7.8 = 18.2$ 3 1 1

with $v[\hat{T}] = N^2\frac{(1-f)}{n}s^2 = \frac{7^2\times 4}{7\times 3}\times\frac{1}{2}\left(2.5^2+3.0^2+2.3^2-\frac{7.8^2}{3}\right)$ 1 2

$= 1.2133$ 2 12

sim. ex. in lecture notes

(c) true value is $2.5+3.2+\ldots\ldots+2.3=20$ 1
as $se[\hat{T}]>1$, then $|20-\hat{T}| < 1.96\times s.e[\hat{T}]$ so it does contain T 1 2

25

Similar example is coursework/textbook

Q2 (a)
$$v[\bar{y}_{st}] = \sum_{h=1}^{H}\frac{N_h^2}{N^2}\underbrace{\frac{(1-n_h/N_h)}{n_h}}_{n_h^{-1}-N_h^{-1}}s_h^2 \qquad \left(\bar{y}_{st} = \frac{1}{N}\sum_{h=1}^{H}N_h\bar{y}_h\right)$$ 5

(b) [use the poststratification which gives the smallest variance estimate] 1

(i) $10^{-4}\times\left(40^2\left(\frac{1}{10}-\frac{1}{40}\right)\times 0.645 + 20^2\left(\frac{1}{7}-\frac{1}{20}\right)\times 0.732 + 10^2\left(\frac{1}{4}-\frac{1}{10}\right)\times 0.708 + 30^2\left(\frac{1}{6}-\frac{1}{20}\right)\times 0.832\right)$ 5
$= \cancel{0.0203}$ (3 sf) 0.02150

(ii) $10^{-4}\times\left(60^2\left(\frac{1}{17}-\frac{1}{60}\right)\times 0.985 + 40^2\left(\frac{1}{10}-\frac{1}{40}\right)\times 1.13\right) = \cancel{0.0300}$ (3sf) 0.02851 5

(iii) $10^{-4}\times\left(50^2\left(\frac{1}{14}-\frac{1}{50}\right)\times 0.854 + 50^2\left(\frac{1}{13}-\frac{1}{50}\right)\times 0.904\right) = 0.0238$ (3sf) 5

(iv) $\left(\frac{1}{27}-\frac{1}{100}\right)\times 1.25 = 0.0338$ (3sf) 3

$\therefore$ Use poststratification by both factors (though the improvement is small over (iii)) 1

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similar example in lectures + notes

lose only two for working out correctly $\hat{R}$ not $\hat{R}$

lose only overall one for working out (correctly) for $\hat{R}$ rather than $\hat{Y}_R$

Q3 (a)

$\bar{X} = \frac{11}{4} = 2.75$

$\bar{Y} = \frac{27}{4} = 6.75$

$R = \frac{6.75}{2.75} = 2.4\dot{5}$

sample labels		$\bar{y}$	$\bar{x}$	$\hat{Y}_R = \bar{X}\frac{\bar{y}}{\bar{x}}$	each row
1	2	5	2.5	5.5	2
1	3	5.5	2.5	6.05	2
1	4	6	2.5	6.60	2
2	3	7.5	3	6.875	2
2	4	8.0	3	$7.\dot{3}$	2
3	4	8.5	3	$7.791\dot{6}$	2

subtract one!

(b)

MEAN $E[\hat{Y}_R] = 6.691\dot{6}$ 2 $\longrightarrow$ $\text{bias}[\hat{Y}_R] = \underline{-0.058\dot{3}}$ 2

$\text{bias}[\hat{R}] = -0.02\dot{1}$

$E[\hat{Y}_R^2] = 45.361$ 2 (5sf)

VAR. $\text{Var}[\hat{Y}_R] = 45.361 - 6.691\dot{6}^2 = \underline{0.58259}$ 2 (5sf)

$\text{Var}[\hat{R}] =$

(c) $\text{Var}[\hat{Y}_R] \doteq \frac{1-f}{n}\left(S_y^2 - 2RS_{xy} + R^2S_x^2\right) = \frac{1}{4}\left(6.91\dot{6} - 2\times 2.\dot{4}\dot{5}\times 1.25 + 2.\dot{4}\dot{5}^2\times 0.25\right)$

value of R 2

$= ~~0.25584~~$ (5sf)

0.57163 2

$\text{Var}[\hat{R}] \doteq 0.075587$

so the error is $\underline{-0.01096}$ 1

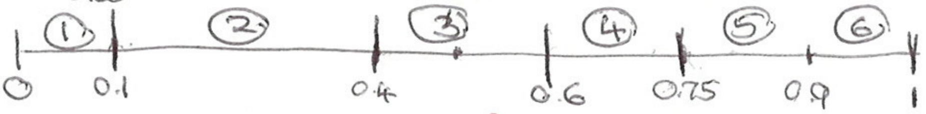
similar examples in lectures + notes

Q4 (a)

label	1	2	3	4	5	6
size	2	6	4	3	3	2
cum size	2	8	12	15	18	20

2

$\longrightarrow$



$u = 0.75\ldots\ldots > 0.75, < 0.9$ so select unit 5 (size 3) 2 1

$u = 0.6\ldots\ldots > 0.6, < 0.75$ so select unit 4 (size 3) 2 1

(b) unbiased estimator is $\tilde{R} = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{x_i}$ 3 $\longrightarrow \tilde{R} = \frac{1}{2}\left(\frac{4}{3} + \frac{6}{3}\right) = 1.\dot{6}$ 3 1

(c) $v[\tilde{R}] = \frac{1}{n}\frac{1}{n-1}\left\{\sum_{i=1}^{n}\left(\frac{y_i}{x_i} - \tilde{R}\right)^2 = \frac{1}{4}\left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right)^2\right.$ 3 3 For $n=2$

$= \frac{1}{4}\left(\frac{4}{3} - \frac{6}{3}\right)^2 = \frac{1}{9} = 0.\dot{1}$ 1

so "95%" C.I is $1.\dot{6} \pm 1.96\sqrt{0.\dot{1}} = \underline{(1.01\dot{3}, 2.32)}$ 2 1

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Q5 (a) $f_U(u) = 1 \quad 0<u<1, \; z.o.$ 1

bookwork

so $f_X(x) = \left|\frac{du}{dx}\right| f_U(u(x))$ 3

$= \lambda e^{-\lambda x}$

$x = -\frac{1}{\lambda}\log u$

$u = e^{-\lambda x}$ 1

$\frac{du}{dx} = -\lambda e^{-\lambda x}$ 1

~~and~~ $[\{0<u<1\} \longleftrightarrow \{0<x<\infty\}$ is $(1,1)]$ 1

so $X \sim Exp(\lambda)$

similar example in lecture

(b) $(\lambda = 0.5)$ 1 $u_1 = .7562$ 1 $\longrightarrow$ $\underline{x_1 = 0.5589}$ 2

$u_2 = .6013$ 1 $\longrightarrow$ $\underline{x_2 = 1.017}$ 2

used in a lab. + checked by simulation

(c) now u_1 (say) $\times 2\pi \longrightarrow \theta = 4.751$ 3 gives a random angle θ in the range $(0, 2\pi)$

and x_2 gives a random $R^2 \sim \chi^2_2$

$r = 1.008$ 3

so $\underline{z = r\cos\theta = 0.03891}$ 2 $\quad \underline{y = r\sin\theta = -1.007}$ 2

gives a pair of realised independently standard normals.

(Box-Müller transformation)

Q6 (marks deducted for poor spelling, English usage, & grammar)

Discussed in class also group proposal essay

Full marks for brief relevant discussion on some or all of the following ~~topics~~

Objectives
Definition of target population
Definition of variables
Methods of measurement & minimising meas. error
Sampling Frame
Pre-test & Pilot Surveys
Non-Response (item & unit, Follow-ups)

2 marks for each (properly expressed) non-overlapping point of relevance.