MAS234 2008 SOLUTIONS

Bookwork

Q1 (a)

$$\bar{y} \pm 1.96\sqrt{\left(1-\frac{n}{N}\right)\frac{s^2}{n}}$$

(b)

Similar question (For n=2 N=4) in coursework and examples in lectures. Also larger examples in labs

allow these two columns to be omitted

or $V[\bar{y}] = 0.3s^2$

prob.	sample $i_1 < i_2$	sample values $y_1\ y_2$	sample mean $\bar{y}$	sample var. $s^2 = \frac{1}{2}(y_1-y_2)^2$	half width ~~or Full CI~~ (3dp) $1.96\sqrt{0.3s^2}$	$(\bar{Y}=3)$ Cover?
1/10	1 2	2, 3	2.5	0.5	0.759	✓
1/10	1 3	2, 5	3.5	4.5	2.277	✓
1/10	1 4	2, 1	1.5	0.5	0.759	✗
1/10	1 5	2, 4	3.0	2.0	1.518	✓
1/10	2 3	3, 5	4.0	2.0	1.518	✓
1/10	2 4	3, 1	2.0	2.0	1.518	✓
1/10	2 5	3, 4	3.5	0.5	0.759	✓
1/10	3 4	5, 1	3.0	8.0	3.036	✓
1/10	3 5	5, 4	4.5	0.5	0.759	✗
1/10	4 5	1, 4	2.5	4.5	2.277	✓
			2	2	2	
			30			

Thus coverage prob. = $8 \times 1/10 = \underline{0.8}$ 2

$$E[\bar{y}] = \frac{1}{10}(2.5 + 3.5 + \cdots + 2.5) = \frac{30}{10} = 3 = \bar{Y} \quad 2$$

so $\bar{y}$ is unbiased for $\bar{Y}$.

MAS 234 2008 SOLUTIONS

bookwork

Q2 (a) $$\bar{y} = \frac{1}{n}\sum_{i=1}^{N} t_i Y_i \qquad \left[\begin{array}{l}\text{where } t_i = 1 \text{ if } i \text{ in sample} \\ \quad\;\; = 0 \text{ otherwise}\end{array}\right]$$

(b) assume $$\left\{\begin{array}{l} E[t_i] = \frac{n}{N} \quad ' \quad Var[t_i] = \frac{n}{N}\left(1-\frac{n}{N}\right) \quad ' \\ Cov[t_i, t_j] = -\frac{n}{N}\frac{1}{N-1}\left(1-\frac{n}{N}\right) \quad ' \end{array}\right.$$

$$E[\bar{y}] = \frac{1}{n}\sum_{i=1}^{N} Y_i E[t_i] = \frac{1}{N}\sum_{i=1}^{N} Y_i = \bar{Y}$$ hence $\bar{y}$ unbiased for $\bar{Y}$ '

bookwork

$$Var[\bar{y}] = \frac{1}{n^2}\left[\sum_{i=1}^{N} Y_i^2 \frac{n}{N}\left(1-\frac{n}{N}\right) - \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} Y_i Y_j \frac{n}{N}\frac{1}{N-1}\left(1-\frac{n}{N}\right)\right]_2$$

$$= \frac{1}{nN}\left(1-\frac{n}{N}\right)\frac{1}{N-1}\left[(N-1)\sum_{i=1}^{N} Y_i^2 - \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} Y_i Y_j\right]$$

$$= \frac{1}{nN}\left(1-\frac{n}{N}\right)\frac{1}{N-1}\left[N\sum_{i=1}^{N} Y_i^2 - \left(\sum_{i=1}^{N} Y_i\right)^2\right] \quad 2$$

$$= \frac{1}{n}\left(1-\frac{n}{N}\right)\underbrace{\frac{1}{N-1}\left[\sum_{i=1}^{N}(Y_i-\bar{Y})^2\right]}_{S^2} = \frac{1-f}{n}S^2 \quad 2$$

MAS 234 2008 SOLUTIONS

Similar example in coursework and lectures

Q3 (a) 3 $\hat{T}_{PS}\left(=\sum_{h=1}^{H} N_h y_h\right) = 248\times 1.85 + 267\times 1.65 = 899.35$ (poststratification estimate of T)

$\hat{Y}_{PS}$: 1.746

$v[\hat{T}_{PS}]\left(=\sum_{h=1}^{H} N_h^2\left(1-\frac{n_h}{N_h}\right)\frac{s_h^2}{n_h}\right)$

5 $= 248^2\left(1-\frac{22}{248}\right)\frac{0.74}{22} + 267^2\left(1-\frac{28}{267}\right)\frac{0.68}{28} = 3435.0$

$v[\hat{Y}_{PS}]$: 0.01295

$s.e. = \sqrt{v[\hat{T}_{PS}]} = 58.609$ (0.1138)

2 $\hat{T}_{PS} \pm 1.96\sqrt{v[\hat{T}_{PS}]} \longrightarrow (784.48, 1014.2)$ (5SF)

(1.523, 1.969)

unseen (in the stratification chapter) but similar considered in the SRS chapter

(b) 2 $\hat{T}(=N\bar{y}) = 515\,\frac{1}{50}(22\times 1.85 + 28\times 1.65)$

$= 895.07$

LOSE 2 MARKS IF ESTIMATED FOR MEAN RATHER THAN TOTAL!

~~LOSE~~

MAS234 2008 SOLUTIONS

very similar question in the test

Q4 (a) $\hat{R} = \bar{y}/\bar{x} = 12/10 = \underline{1.2}$

$$v[\hat{R}] = \frac{1-f}{n\bar{X}^2}\left(s_y^2 - 2\hat{R}s_{xy} + \hat{R}^2 s_x^2\right) \qquad \bar{X} = \frac{240}{100} = 2.4$$

$$= \frac{1-4/100}{4\times 2.4^2}\left(2 + 2.4\times 0.25 + 1.2^2\times 1.\dot{6}\right)$$

$$= \underline{0.208\dot{3}}$$

(b) $\bar{y}/\bar{X} = 3/2.4 = \underline{1.25}$

$$v[\bar{y}/\bar{X}] = \frac{1-f}{n\bar{X}^2}s_y^2 = \underline{0.08\dot{3}}$$

(c) $\tilde{R} = \frac{1}{4}\left(\frac{5}{2} + \frac{2}{1} + \frac{3}{3} + \frac{2}{4}\right) = \cancel{1.25}$ 1.5

$$v[\tilde{R}] = \frac{1}{4}\cdot\frac{1}{3}\left(\left(\frac{5}{2}\right)^2 + 2^2 + 1^2 + 0.5^2 - 5^2/4\right) = \frac{5}{24}$$

$= \cancel{0.1875}$ $0.208\dot{3}$

MAS 234 2008 solutions

Q5 (a) $f_U(u) = 1 \quad 0<u<1$, zero otherwise

bookwork

so $f_X(x) = \left|\frac{du}{dx}\right| f_U(u(x))$ $\qquad x = -\frac{1}{\theta}\log u$

$= \theta e^{-\theta x}$ $\qquad u = e^{-\theta x}.$

$\frac{du}{dx} = -\theta e^{-\theta x}$

and $\{0<u<1\} \longleftrightarrow \{0<x<\infty\}$

so X has exponential pdf with par. θ

unseen application of transformation lemma in lectures

(b) $f_X(x) = \left|\frac{dy}{dx}\right| f_Y(y(x))$ $\qquad x = \log(y/y_0)$

$y = y_0 e^x$

$\frac{dy}{dx} = y_0 e^x$

$= \cancel{y_0 e^x} \cdot \theta y_0^{\theta} \cancel{y_0}^{-(\theta+1)} e^{-\theta x} \cancel{e^{-x}}$

$= \theta e^{-\theta x}$

so conclusion as before.

method described in lectures and implemented in labs.

(c) [either write $Y = y_0 e^X = y_0 e^{-\frac{1}{\theta}\log U} = y_0 U^{-1/\theta}$]

or $Y = y_0 e^X$, $X = -\frac{1}{\theta}\log U$

Generate $u_1, u_2, \ldots$ (to any required degree of accuracy) by taking consecutive strings of say r digits as decimal numbers in $0,1$

[eg $(3658)(7012)\ldots \longrightarrow u_1 = 0.3658$, $u_2 = 0.7012$ etc.]

then apply above trans. to generate $x_1, x_2, \ldots$

& hence $y_1, y_2, \ldots$

MAS234 2008 SOLUTIONS.

Q6

(marks deducted for poor spelling, grammar & English usage)

Full marks for brief, relevant discussion on some or all of the following topics:

Objectives
Definition of target population
Definition of variables
Methods of measurement & minimising error
Sampling frame
Pretest, Pilot surveys
Non response (item & unit, follow up strategy)

Discussed in class
group proposal essay
for group project
tackled similar issues