QUEEN MARY, UNIVERSITY OF LONDON

MAS 417 / MTHM028 — Association Schemes and Partially Balanced Designs

Assignment 1 — **For handing in on 6 February 2006**

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1 Explain why the parameters of an association scheme satisfy

(a) $p^j_{0j} = 1$;

(b) $p^k_{ij} = p^k_{ji}$.

(c) if $i \neq j$ then $p^0_{ij} = p^i_{0j} = p^j_{i0} = 0$.

2 Verify that a graph is strongly regular if and only if it is neither complete nor null and the sets of edges and non-edges form an association scheme on the set of vertices. Relate the parameters of the association scheme to those of the strongly regular graph.

3 Draw a finite graph that is regular but not strongly regular.

4 Let A and B be matrices in $F^{\Gamma\times\Delta}$ and $F^{\Delta\times\Phi}$ respectively, where F is a field and Γ, Δ and Φ are finite sets. Prove that $(AB)' = B'A'$.

5 Let A be a symmetric matrix with zero diagonal whose entries are 0 and 1. Suppose that there are integers x, y and z such that $A^2 = xI + yA + zJ$. Show that A, I and $J - A - I$ are the adjacency matrices of an association scheme, and find its parameters.

6 Let Ω consist of the twelve faces of the regular dodecahedron. For α and β in Ω, let α and β be

- first associates if faces α and β have an edge in common;
- third associates if faces α and β are antipodal;
- second associates otherwise (if $\alpha \neq \beta$).

Show that this defines an association scheme on Ω and find its parameters.

7 Two Latin squares of the same size are said to be *orthogonal to each other* if each letter of one square occurs exactly once in the same position as each letter of the second square. A collection of Latin squares of the same size is said to be *mutually orthogonal* if every pair of squares in it is orthogonal.

Suppose that $\Lambda_1, \ldots, \Lambda_r$ is a set of r mutually orthogonal Latin squares of size n. Let Ω be the set of n^2 cells in the array. For distinct α, β in Ω, let α and β be first associates if α and β are in the same row or are in the same column or have the same letter in any of $\Lambda_1, \ldots, \Lambda_r$; otherwise α and β are second associates.

(a) Find the size of $\mathcal{C}_1(\alpha)$. Hence find an upper bound on r.

(b) Show that these definitions of $\mathcal{C}_1$ and $\mathcal{C}_2$ make an association scheme on Ω. It is called the *Latin-square type* association scheme $\mathrm{L}(r+2, n)$. When does it have only one associate class?

8 Find the parameters of the Johnson scheme $\mathrm{J}(7,3)$.

9 Find the parameters of the Hamming scheme $\mathrm{H}(5,4)$.