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M. Sci. Examination 2003

MAS417 Association Schemes and Partially Balanced Designs

Duration: 3 hours

Date and time: 29 May 2003, 1000h

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Question 1 Describe the following association schemes, and find their character tables.

(a) $\underline{\underline{4}}$; [3 + 3]

(b) $\mathcal{L}$, which is of Latin-square type $\mathrm{L}(3,7)$; [3 + 10]

(c) $\mathcal{L} \times \underline{\underline{4}}$, where $\mathcal{L}$ is as above. [3 + 3]

Question 2 (a) Given an incomplete-block design for treatment set Θ, state what it means for the design to be *partially balanced*. [3]

(b) Prove that, for a partially balanced incomplete-block design, the variance of the simple contrast for the difference between treatments θ and η is a function of the associate class containing (θ, η). [7]

(c) Show that the following design for $\Theta = \{A, B, C, D, E, F\}$ is partially balanced. Find its canonical efficiency factors and the average variance of simple contrasts when the variance of each response is σ^2.

$$\{A, C, F\},\ \{A, D, E\},\ \{B, C, E\},\ \{B, D, F\}.$$

[15]

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Question 3 (a) Define a *strongly regular graph*. [2]

(b) Let $\mathcal{G}$ be a strongly regular graph with 21 vertices and valency a. Prove that a must be even. [5]

(c) For a equal to each of 2, 4, 6, 8, 10, *either* give an example of a strongly regular graph on 21 vertices with valency a *or* prove that no such strongly regular graph exists. [18]

Question 4 (a) Define the *Johnson* association scheme J(6, 3) (you may assume that it is an association scheme). Show that one of its associate classes defines a distance-regular graph and find the parameters of this distance-regular graph. [13]

(b) For $i = 0, \ldots, 3$, let A_i be the adjacency matrix for vertices at distance i in the above graph. For $i = 0, \ldots, 3$, express A_iA_1 as a linear combination of the adjacency matrices. Hence find the eigenvalues of A_1. [12]

Question 5 (a) Let Ω be a finite set. Explain what an *orthogonal block structure* on Ω is. [7]

(b) Suppose that $\mathcal{F}$ is an orthogonal block structure on Ω. For each F in $\mathcal{F}$, define the subset $\mathcal{C}_F$ of $\Omega \times \Omega$ to consist of all those pairs (α, β) such that α and β are in the same class of F but there is no G in $\mathcal{F}$ such that G is finer than F and α and β are in the same class of G. Prove that the non-empty classes $\mathcal{C}_F$ form an association scheme on Ω. [18]

Question 6 (a) Define a *blueprint* for $\mathbb{Z}_n$. [5]

(b) Show that $\{0\}$, $\{1, 3, 4, 9, 10, 12\}$ and $\{2, 5, 6, 7, 8, 11\}$ is a blueprint for $\mathbb{Z}_{13}$. [7]

(c) In $\mathbb{Z}_{13}$, let $\Phi = \{0, 3, 4\}$. Find the concurrence matrix for the cyclic design generated by Φ. Hence find the variances of simple contrasts when the variance of each response is σ^2. [13]

Question 7 (a) Explain what is meant by the *character table* C of an association scheme whose adjacency matrices are $A_0, A_1, \ldots, A_s$. What is the significance of the matrix D, where $D = C^{-1}$? [5]

(b) State and prove the orthogonality relations for C. [14]

(c) State and prove the orthogonality relations for D. [6]

 END OF EXAMINATION