Chapter 3

Partially Balanced Incomplete-Block Designs

3.1 Block designs

Let Ω be a set of experimental units, called ‘plots’, to which we can apply treatments in an experiment. Suppose that Ω is partitioned into b blocks of k plots each. Define $B$ in $\mathbb{R}^{\Omega \times \Omega}$ by

$$B(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same block} \\ 0 & \text{otherwise.} \end{cases}$$

Then the group divisible association scheme GD$(b, k)$ on Ω has adjacency matrices $I$, $B - I$ and $J - B$.

A block design on Ω for treatment-set Θ is a function $\phi: \Omega \rightarrow \Theta$. Strictly speaking, the design is the quadruple $(\Omega, Q, \Theta, \phi)$, where Ω is the set of plots, $Q$ is the group divisible association scheme on Ω, Θ is the set of treatments and $\phi$ is the design function from Ω to Θ. We say that “treatment $\theta$ occurs on plot $\omega$” if $\phi(\omega) = \theta$. Define the design matrix $X$ in $\mathbb{R}^{\Omega \times \Theta}$ by

$$X(\omega, \theta) = \begin{cases} 1 & \text{if } \phi(\omega) = \theta \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Delta$ be the set of blocks. The incidence matrix is the matrix $N$ in $\mathbb{R}^{\Delta \times \Theta}$ given by

$$N(\delta, \theta) = |\{\omega \in \delta : \phi(\omega) = \theta\}|.$$

The design is binary if $N(\delta, \theta) \in \{0, 1\}$ for all $\delta$ in $\Delta$ and all $\theta$ in $\Theta$. In this case we often identify block $\delta$ with $\{\theta \in \Theta : N(\delta, \theta) = 1\}$ and identify the design with the family of blocks.
The design is a complete-block design if it is binary and \( k = |\Theta| \), so that \( N = J_{\Delta,\Theta} \). It is an incomplete-block design if it is binary and \( k < |\Theta| \), so that each block is incomplete in the sense of not containing all of the treatments.

When Yates introduced incomplete-block designs he called them ‘incomplete block designs’ to contrast them with complete block designs. However, when pure mathematicians took up his term, they assumed that ‘incomplete’ referred to the design and meant that not every \( k \)-subset of the treatments was a block. That is why I insist on the hyphen in ‘incomplete-block design’.

The replication \( r_\theta \) of treatment \( \theta \) is
\[
|\{ \omega \in \Omega : \phi(\omega) = \theta \}|.
\]
The design is equi-replicate if all treatments have the same replication. If there are \( t \) treatments with replication \( r \) then
\[
tr = bk. \tag{3.1}
\]
In such a design, \( X'X = rI \).

The trick for remembering the parameters is that \( b \) and \( k \) are to do with blocks while \( t \) and \( r \) concern treatments. (In his first paper on incomplete-block designs, Yates was considering agricultural field trials to compare different varieties. So he called the treatments varieties, and let their number be \( v \). This gives the mnemonic \( va'rites \). In his second paper on the subject he switched to the less specific word treatment, and established the notation \( t, r, b, k \) used here. Statisticians have tended to follow this second usage, but pure mathematicians have retained \( v \) for the number of treatments, even though they often call them points.)

The concurrence \( \Lambda(\theta, \eta) \) of treatments \( \theta \) and \( \eta \) is
\[
|\{ (\alpha, \beta) \in \Omega \times \Omega : \alpha, \beta \text{ in the same block, } \phi(\alpha) = \theta, \phi(\beta) = \eta \}|.
\]
The matrix \( \Lambda \) is called the concurrence matrix. Equi-replicate block designs in which every non-diagonal concurrence is in \{0, 1\} are called configurations.

**Lemma 3.1**

(i) If the design is binary then \( \Lambda(\theta, \theta) \) is equal to the replication of \( \theta \), and, for \( \eta \neq \theta \), \( \Lambda(\theta, \eta) \) is equal to the number of blocks in which \( \theta \) and \( \eta \) both occur.

(ii) \( \sum_{\eta \in \Theta} \Lambda(\theta, \eta) = k r_\theta \); in particular, in an equi-replicate design
\[
\sum_{\eta \in \Theta, \eta \neq \theta} \Lambda(\theta, \eta) = r(k - 1).
\]

(iii) \( \Lambda = X'BX = N'N \).
Example 3.1 A small design on 12 experimental units is shown below. The set $\Omega$ consists of three blocks $\delta_1$, $\delta_2$, $\delta_3$ of four plots each, so $b = 3$ and $k = 4$. The treatment-set $\Theta$ is $\{1, 2, 3, 4, 5, 6\}$, so $t = 6$. The design is equi-replicate with $r = 2$.

$$
\begin{array}{cccccccccccc}
\Omega & \delta_1 & \delta_2 & \delta_3 \\
\omega_1 & 1 & 2 & 3 & 4 & 1 & 2 & 5 & 6 & 3 & 4 & 5 & 6 \\
\end{array}
$$

With the plots in the obvious order, the blocks matrix $B$ is given by

$$
B = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
$$

The stratum projectors are $\frac{1}{12}J$, $\frac{1}{4}B - \frac{1}{12}J$ and $I - \frac{1}{4}B$.

With the plots in the same order as above and treatments also in the obvious order, the design matrix $X$ is given by

$$
X = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

Then $X'X = 2I$. 

CHAPTER 3. PARTIALLY BALANCED INCOMPLETE-BLOCK DESIGNS

The incidence matrix $N$ and concurrence matrix $\Lambda$ are shown with their rows and columns labelled for clarity.

\[
N = \begin{bmatrix}
\delta_1 & 1 & 2 & 3 & 4 & 5 & 6 \\
\delta_2 & 1 & 1 & 1 & 1 & 0 & 0 \\
\delta_3 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

\[
\Lambda = \begin{bmatrix}
1 & 2 & 2 & 1 & 1 & 1 \\
2 & 2 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 2 & 2 & 1 & 1 \\
4 & 1 & 1 & 2 & 2 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 2 & 2 \\
6 & 1 & 1 & 1 & 1 & 2 & 2 \\
\end{bmatrix}
\]

It is clear that $\Lambda = N'N = X'BX$.

The dual block design to $(\Omega, Q, \Theta, \phi)$ is obtained by interchanging the roles of $\Theta$ and $\Delta$. So the experimental units of the dual design are still the elements of $\Omega$, but the blocks are the sets $\phi^{-1}(\theta)$ for $\theta$ in $\Theta$. These new blocks define a group divisible association scheme $Q^*$ on $\Omega$ of type GD$(t, r)$. The dual design is the quadruple $(\Omega, Q^*, \Delta, \psi)$ where the dual design function $\psi: \Omega \rightarrow \Delta$ is given by $\psi(\omega) =$ old block containing $\omega$.

The incidence matrix for the dual design is $N'$.

**Example 3.2** The dual of the design in Example 3.1 has treatment-set $\{\delta_1, \delta_2, \delta_3\}$. Its blocks are $\{\delta_1, \delta_2\}$, $\{\delta_1, \delta_3\}$, $\{\delta_1, \delta_3\}$, $\{\delta_2, \delta_3\}$ and $\{\delta_2, \delta_3\}$.

**Definition** The treatment-concurrence graph of an incomplete-block design is the (not necessarily simple) graph whose vertices are the treatments and in which the number of edges from $\theta$ to $\eta$ is $\Lambda(\theta, \eta)$ if $\theta \neq \eta$.

**Definition** An incomplete-block design is connected if its treatment-concurrence graph is connected.