Queen Mary
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MAS 305 Algebraic Structures II

Notes 9 **Autumn 2006**

Composition series and soluble groups

Definition A normal subgroup N of a group G is called a *maximal normal subgroup* of G if

(a) $N \neq G$;

(b) whenever $N \leqslant M \trianglelefteq G$ then either $M = N$ or $M = G$.

By the Correspondence Theorem, if $N \lhd G$ and $N \neq G$ then every normal subgroup of G/N corresponds to a normal subgroup of G containing N. So a normal subgroup N is maximal if and only if G/N is simple.

Definition Given a group G, a *composition series* for G of length n is a sequence of subgroups

$$G = B_0 > B_1 > \cdots > B_n = \{1_G\}$$

such that

(a) $B_i \lhd B_{i-1}$ for $i = 1, \ldots, n$,

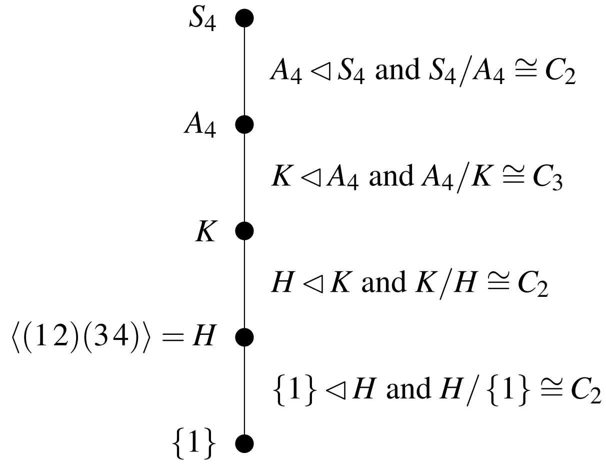
(b) B_{i-1}/B_i is simple for $i = 1, \ldots, n$.

In particular, B_1 is a maximal normal subgroup of G and B_{n-1} is simple. The (isomorphism classes of the) quotient groups B_i/B_{i-1} are called *composition factors* of G.

Example S_4 has the following composition series of length 4, where K is the Klein group $\{(1),\ (1\,2)(3\,4),\ (1\,3)(2\,4),\ (1\,4)(2\,3)\}$.

$$S_4 > A_4 > K > \langle (1\,2)(3\,4) \rangle > \{1\}$$

We know that $A_4 \lhd S_4$; the composition factor $S_4/A_4 \cong C_2$. We have seen that $K \lhd A_4$; and $A_4/K \cong C_3$. All subgroups of K are normal in K, because K is Abelian. Both $K/\langle (1\,2)(3\,4) \rangle$ and $\langle (1\,2)(3\,4) \rangle/\{1\}$ are isomorphic to C_2. So the composition factors of S_4 are C_2 (three times) and C_3 (once).



Example If G is simple then its only composition series is $G > \{1\}$, of length 1.

Example $(\mathbb{Z},+)$ has no composition series. If $H \leqslant \mathbb{Z}$ then H is cyclic of infinite order. If $H = \langle x \rangle$ then $\langle 2x \rangle$ is a subgroup of H with $\{0\} \neq \langle 2x \rangle \neq H$, and $\langle 2x \rangle \lhd H$ because H is Abelian. So H is not simple. If $B_0 > B_1 > \cdots > B_n$ is a composition series then B_{n-1} is simple, so there can be no composition series.

Theorem Every finite group G has a composition series.

Proof We use induction on $|G|$. If $|G| = 1$ then the composition series is just $G = B_0 = \{1\}$.

Assume that $|G| > 1$ and that the result is true for all groups of order less than $|G|$. Since G is finite, G has at least one maximal normal subgroup N. Then $|N| < |G|$, so by induction N has a composition series $N = B_1 > B_2 > \cdots > B_n = \{1\}$ with $B_i \lhd B_{i-1}$ and B_{i-1}/B_i simple for $i = 2, \ldots, n$. Putting $B_0 = G$ gives the composition series $G = B_0 > B_1 > \cdots > B_n = \{1\}$ for G, because $B_1 \lhd B_0$ and $B_0/B_1 = G/N$, which is simple. $\square$

The next theorem shows that statements such as "the composition factors of S_4 are C_2 (three times) and C_3" do not depend on the choice of composition series.

Jordan-Hölder Theorem Suppose that the finite group G has two composition series

$$G = B_0 > B_1 > \cdots > B_n = \{1\}$$

and

$$G = C_0 > C_1 > \cdots > C_m = \{1\}.$$

Then $n = m$ and the lists of composition factors for the two series are identical in the sense that if $|H| \leqslant |G|$ and

$$\phi(H) = |\{i \geqslant 1 : B_{i-1}/B_i \cong H\}|$$

and

$$\psi(H) = |\{i \geqslant 1 : C_{i-1}/C_i \cong H\}|$$

then $\phi(H) = \psi(H)$.

Proof We use induction on $|G|$. The result is true if $|G| = 1$.

Assume that $|G| > 1$ and that the result is true for all groups of order less than $|G|$. Then n and m are both positive. Put

$$\phi_1(H) = |\{i \geqslant 2 : B_{i-1}/B_i \cong H\}|$$

and

$$\psi_1(H) = |\{i \geqslant 2 : C_{i-1}/C_i \cong H\}|.$$

Then

$$\phi(H) = \begin{cases} \phi_1(H) + 1 & \text{if } H \cong G/B_1 \\ \phi_1(H) & \text{otherwise} \end{cases}$$

and

$$\psi(H) = \begin{cases} \psi_1(H) + 1 & \text{if } H \cong G/C_1 \\ \psi_1(H) & \text{otherwise.} \end{cases}$$

First suppose that $B_1 = C_1$. Then B_1 has the following two composition series:

$$B_1 > \cdots > B_n = \{1\}$$

of length $n - 1$, and

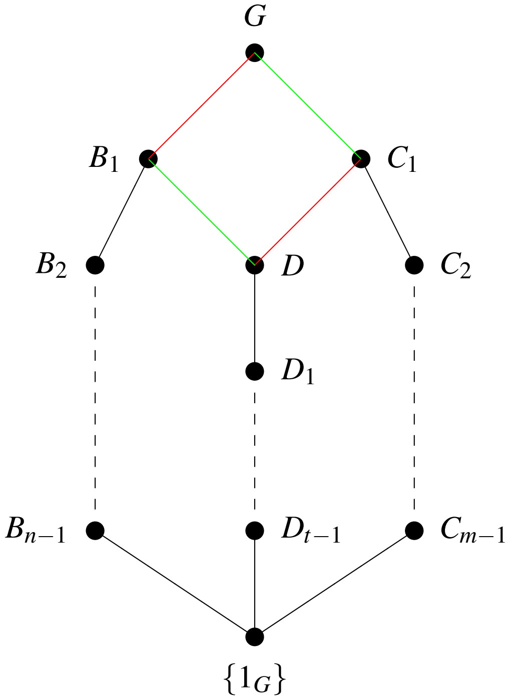
$$B_1 = C_1 > \cdots > C_m = \{1\}$$

of length $m - 1$. Now, $|B_1| < |G|$, so by the inductive hypothesis the result is true for B_1, so $n = m$ and $\phi_1(H) = \psi_1(H)$ for all H. If $H \cong G/B_1 = G/C_1$ then $\phi(H) = \phi_1(H) + 1 = \psi_1(H) + 1 = \psi(H)$; otherwise $\phi(H) = \phi_1(H) = \psi_1(H) = \psi(H)$. Therefore the result is true for G.

Secondly, suppose that $B_1 \neq C_1$, and put $D = B_1 \cap C_1$. Because B_1 and C_1 are both normal subgroups of G, so is B_1C_1. If $B_1C_1 = B_1$ then $B_1 > C_1$, but this cannot be true, because C_1 is a *maximal* normal subgroup of G. Hence $B_1 \leqslant B_1C_1 \trianglelefteq G$ and $B_1C_1 \neq B_1$, so $B_1C_1 = G$. By the Third Isomorphism Theorem, $G/B_1 = B_1C_1/B_1 \cong C_1/B_1 \cap C_1 = C_1/D$. Similarly, $G/C_1 \cong B_1/D$.

Let $D = D_0 > D_1 > \cdots > D_t = \{1\}$ be a composition series for D, and put

$$\theta(H) = |\{i \geqslant 1 : D_{i-1}/D_i \cong H\}|.$$



Now C_1/D is simple, so

$$C_1 > D > D_1 > \cdots > D_t = \{1\}$$

is a composition series for C_1. So is

$$C_1 > C_2 > \cdots > C_m = \{1\}.$$

But $|C_1| < |G|$, so by inductive hypothesis $t + 1 = m - 1$ and

$$\psi_1(H) = \begin{cases} \theta(H) + 1 & \text{if } H \cong C_1/D \\ \theta(H) & \text{otherwise.} \end{cases}$$

Applying the similar argument to B_1 gives $t+1 = n-1$ and

$$\phi_1(H) = \begin{cases} \theta(H)+1 & \text{if } H \cong B_1/D \\ \theta(H) & \text{otherwise.} \end{cases}$$

Hence $n = m$. Moreover, since $G/B_1 \cong C_1/D$ and $G/C_1 \cong B_1/D$, either

(a) $G/B_1 \cong G/C_1$ and $\phi(H) = \psi(H) = \begin{cases} \theta(H)+2 & \text{if } H \cong G/B_1 \\ \theta(H) & \text{otherwise,} \end{cases}$ or

(b) $G/B_1 \ncong G/C_1$ and $\phi(H) = \psi(H) = \begin{cases} \theta(H)+1 & \text{if } H \cong G/B_1 \text{ or } H \cong G/C_1 \\ \theta(H) & \text{otherwise.} \end{cases}$ □

Definition A finite group is *soluble* if all its composition factors are cyclic of prime order.

Example S_4 is soluble.

Example S_5 is not soluble, because its only composition series is $S_5 > A_5 > \{1\}$.

We have already shown that if $|G| = p^n$ for some prime p then G has subgroups

$$\{1_G\} = G_0 < G_1 < \cdots < G_n = G$$

with $G_i \trianglelefteq G$ and $|G_i| = p^i$ for $i = 0, \ldots, n$. So $|G_{i+1}/G_i| = p$ so $G_{i+1}/G_i \cong C_p$ for $i = 0, \ldots, n-1$. Thus every finite p-group is soluble.

A composition series in which every subgroup is normal in the whole group is called a *chief* series. A finite group is *supersoluble* if it has a chief series all of whose composition factors are cyclic of prime order. So all finite p-groups are supersoluble.

Theorem If H is a normal subgroup of a finite group G, and if H and G/H are both soluble then G is soluble.

Proof Let $H = H_0 > H_1 > \cdots > H_r = \{1\}$ be a composition series for H. Let $G/H = K_0 > K_1 > \cdots > K_s = \{H\}$ be a composition series for G/H. By the Correspondence Theorem, there are subgroups $G_0, \ldots, G_s$ of G containing H such that $G_i/H = K_i$ for $i = 0, \ldots, s$ and $G_i \lhd G_{i-1}$ for $i = 1, \ldots, s$. By the Second Isomorphism Theorem,

$$K_{i-1}/K_i = (G_{i-1}/H)/(G_i/H) \cong G_{i-1}/G_i.$$

Then

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_s = H = H_0 \rhd H_1 \rhd \cdots \rhd H_r = \{1\}$$

is a composition series for G in which every composition factor is cyclic of prime order. □

This proof also shows that if $H \lhd G$ and H has a composition series of length r and G/H has a composition series of length s then G has a composition series of length $r+s$. In other words, (the composition length of G) $=$ (the composition length of H) $+$ (the composition length of G/H).

Corollary All finite Abelian groups are soluble.

Proof Use induction on $|G|$. If $|G| = 1$ then G is soluble.

Assume that G is Abelian, that $|G| > 1$ and that all Abelian groups of order less than $|G|$ are soluble. By Cauchy's Theorem, G contains a subgroup H of prime order. Thus H is soluble. Since G is Abelian, $H \unlhd G$ and G/H is Abelian. But $|H| > 1$ so $|G/H| < |G|$, so G/H is soluble, by inductive hypothesis. By the preceding theorem, G is soluble. $\square$

Theorem Let G be a finite group. Then G is soluble if and only if there is a sequence of subgroups

$$G = B_0 > B_1 > \cdots > B_n = \{1\}$$

such that

(a) $B_i \lhd B_{i-1}$ for $i = 1, \ldots, n$

(b) B_{i-1}/B_i is Abelian for $i = 1, \ldots, n$.

Proof If G is soluble then any composition series satisfies (a) and (b) with each B_{i-1}/B_i cyclic of prime order, hence Abelian.

Conversely, use induction on the order of G. The result is true if $|G| = 1$. Now assume that $|G| > 1$ and that the result is true for all groups of smaller order. Suppose that G has a such a sequence. Then B_{i-1}/B_i is Abelian for $i = 2, \ldots, n$, so B_1 satisfies the conditions. Also, $|B_1| < |G|$. By inductive hypothesis, B_1 is soluble, Moreover, $B_1 \lhd G$ and G/B_1 is Abelian, hence soluble, by the preceding corollary. Hence G is soluble, by the preceding theorem. $\square$