Scenes from mathematical life

Peter J. Cameron

Forder lectures
April 2008
Never apologize, always explain: Scenes from mathematical life

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March 2008
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The Steward lectures

▶ Lecture 1: Before and beyond Sudoku
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- Lecture 2: Proving Theorems in Tehran
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‘I count a lot of things that there’s no need to count,’ Cameron said. ‘Just because that’s the way I am. But I count all the things that need to be counted.’
From Higman–Sims to Urysohn

Mathematicians in Scandale
My 60th birthday card (by Neill Cameron)
The adjacency matrix of a graph has rows and columns indexed by the vertices of the graph; the entry in position \((v, w)\) is 1 if \(v\) is joined to \(w\), 0 otherwise.

\[
\begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\]
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The spectrum of a graph

The **spectrum** of a graph is the spectrum (the multiset of eigenvalues) of its adjacency matrix.
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What does the spectrum tell us about the graph?
Graphs with least eigenvalue $-2$

Two classes were known:

- **Line graphs** (vertices of $L(\Gamma)$ are edges of $\Gamma$, joined if they meet in a vertex);

- **Cocktail party graphs** (vertices paired up, each vertex joined to every other except its pair).
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Hoffman merged these two classes together to obtain generalized line graphs.
The theorem

Hoffman (unpublished) may have showed that any “sufficiently large” connected graph with least eigenvalue $-2$ is a generalized line graph. No indication what “sufficiently large” meant.
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In the 1970s, Jaap Seidel (Eindhoven) and Jean-Marie Goethals (Brussels) were working on this when I visited them.
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**Root systems**

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A root system is **indecomposable** if it is not contained in the union of two non-zero orthogonal subspaces; it is **spherical** if all roots have the same length.
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In the course of their classification of simple Lie algebras over the complex numbers, Cartan and Killing had to find all the indecomposable root systems. The spherical ones, which concern us here, form two infinite families, $A_n$ (for $n \geq 1$) and $D_n$ (for $n \geq 4$), and three “sporadic” ones, $E_6$, $E_7$ and $E_8$. (The subscript is the dimension of the Euclidean space.)
The root systems $A_2$ and $A_3$
The connection

Let $\Gamma$ have adjacency matrix $A$ with least eigenvalue $-2$. Then $2I + A$ is \textbf{positive semi-definite}, so is a matrix of inner products of a set of vectors in Euclidean space. The lines spanned by these vectors make angles $90^\circ$ or $60^\circ$ with one another.
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So the graph can be “embedded” in $A_n, D_n, E_6, E_7$ or $E_8$. 
The result

Since $A_n$ is contained in $D_{n+1}$ and the exceptions in $E_8$, we only need consider $D_n$ and $E_8$. 
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But the story is not over …
Möbius function

This is a generalization of the “Inclusion–Exclusion Principle”.

If we know the size of the whole set, and the sizes of the circles and their intersections, we can calculate the size of the part outside all the circles. It is a sum of the other numbers multiplied by $+1$ or $-1$. 
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For more general situations, we replace the $\pm 1$s by the values of the Möbius function.
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IPM grounds
The mountains from IPM
Details of the museums of Tehran (to which we were taken on excursions)
Daily News

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- Competitions for students, e.g. “Discover the middle names of the invited speakers”
The winner ...
Proving theorems in Tehran

During and after the conference I did three pieces of work which resulted in papers in the conference proceedings.

One of these was work with three postdocs at IPM: Maimani, Omidi and Tayfeh-Reziae, in connection with a problem in design theory. We have a particular permutation group acting on a set of $n$ elements. (Actually the group $\text{PSL}(2, q)$, where $n = q + 1$). We want to find, for each value of $k$, all possible sizes of sets of $k$-element subsets which admit the action of this group, and how many of each size there are.
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To solve this problem, we need to know three things about the group:

▶ All of its subgroups (these were determined by Dickson in the early 20th century).
▶ Their orbit lengths (these are relatively easy and were worked out before).
▶ The so-called “Möbius function” of each possible subgroup. This turns out also to be known but is more obscure.
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Ignoring signs, the values of the Möbius function of these three groups turn out to be 3, 2, 1 respectively (ignoring signs).

But there is another occurrence of these numbers …
John McKay’s most famous discovery was

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This was at the time when the Monster, the largest sporadic finite simple group, had been “discovered” but not constructed. Evidence suggested that the smallest size of matrices which can represent this group over the complex numbers is $196{,}883 \times 196{,}883$. 

The number 196 884 is the first non-trivial Fourier coefficient of the modular function, which arises in classical (nineteenth-century) complex analysis. At the time McKay was maybe the only mathematician in the world who knew both of these facts. This led to the conjectures termed “Monstrous moonshine” by Conway and Norton and proved by Borcherds, connections to conformal field theory and Lie algebras, etc.
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The McKay correspondence

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Each of these groups is described by a graph, whose vertices are the irreducible representations of the group, vertices $V$ and $W$ being joined if $W$ is a constituent of $V \otimes S$, where $S$ is the representation by $2 \times 2$ matrices we start with.
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The fact that $S$ is unitary implies that the graph is undirected. If we label each vertex with its dimension, the number at each vertex is the sum of the numbers at its neighbours.
The graphs associated to the binary polyhedral groups are precisely the extended Coxeter–Dynkin diagrams associated with the exceptional root systems $E_6$, $E_7$ and $E_8$. These diagrams are obtained by taking a “fundamental basis” (with non-positive inner products), and adjoining the “largest root”.

![Diagram of extended Coxeter–Dynkin diagrams for $E_6$, $E_7$, and $E_8$.]
Connection numbers

Each root system spans a lattice $L$ in Euclidean space. Because the inner products of root vectors are integers, the lattice is contained in its dual lattice $L^\dagger$, consisting of all vectors $v$ such that $v \cdot w \in \mathbb{Z}$ for all $w \in L$. It is known that $L^\dagger / L$ is a finite group. Its order is the connection number of the root system.
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What is the connection?
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Its automorphism group has a simple subgroup of index 2, the sporadic Higman–Sims group.
Henson’s graph

Henson discovered an infinite graph with some similarities. It has no triangles, and the automorphism group is transitive on $n$-claws for all $n$. (Its automorphism group is also simple, a very recent result of Macpherson and Tent.)
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For rather complicated reasons I began to wonder: Does Henson’s graph admit a cyclic automorphism? That is, can you arrange the vertices along a line so that a shift one place to the right preserves the graph?
Sum-free sets

If so, label the vertices by integers, and let $S$ be the set of positive neighbours of zero. Then
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- the graph is Henson’s if and only if the sum-free set is universal.

No explicit construction of a universal sum-free set is known …
If we forget about the triangle-free condition, there is a remarkable countable graph called the random graph. As its name suggests, if you choose edges at random, you are almost certain to get this graph!
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If you choose a set $S$ of positive integers at random, you are almost certain to get a universal set, which will give a cyclic automorphism of the random graph.
Random sum-free sets

If you choose a sum-free set at random, it turns out that you don’t get a universal sum-free set. Something much more interesting happens …
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Universal sum-free sets do exist; the existence proof is non-constructive, but uses ideas from topology (Baire category) rather than probability.
The Urysohn space

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By similar methods, we were able to show that the Urysohn space also admits a cyclic isometry all of whose cycles are dense; so the space has an abelian group structure. Indeed it has many different abelian group structures! The story goes on …