Combinatorial representations

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A matroid can be described in many different ways: by the independent sets, the bases, the minimal dependent sets, the rank function ...

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- Graphic matroids: E is the edge set of a graph, and a set of edges is independent if it contains no circuit (so that the rank of a set A of edges is $n - c(A)$, where n is the number of vertices and $c(A)$ the number of connected components in the graph with edge set A.

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- Transversal matroids: E is the index set of a family of subsets of a set S, and a subset of E is independent if the corresponding subfamily possesses a transversal.

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I will use bases to describe matroids and similar structures in this talk.

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Let E be the ground set and $\mathcal{B}$ the family of bases of a matroid M of rank r. A vector representation of M is an assignment of a vector $v_i \in F^r$ to each $i \in E$, such that, for $i_1, \ldots, i_r \in E$,

$$(v_{i_1}, \ldots, v_{i_r}) \text{ is a basis for } F^r \Leftrightarrow \{i_1, \ldots, i_r\} \in \mathcal{B}.$$

. . . in dual form

Now regard the representing vectors $v_1, \ldots, v_r$ as lying in the dual space of F^r. To emphasise this I will write f_i instead of v_i; thus f_i is a function from F^r to F.

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Notation: if $f_{i_1}, \ldots, f_{i_r} : F^r \to F$, then we regard the r-tuple $(f_{i_1}, \ldots, f_{i_r})$ as being a function from F^r to F^r.

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Notation: if $f_{i_1}, \ldots, f_{i_r} : F^r \to F$, then we regard the r-tuple $(f_{i_1}, \ldots, f_{i_r})$ as being a function from F^r to F^r.
Now a vector representation of the matroid M is an assignment of a linear map $f_i : F^r \to F$ to each $i \in E$, so that

$$(f_{i_1}, \ldots, f_{i_r}) : F^r \to F^r \text{ is a bijection } \Leftrightarrow \{i_1, \ldots, i_r\} \in \mathcal{B}.$$

. . . generalised

Let $\mathcal{B}$ be any family of r-subsets of a ground set E, and let A be an alphabet of size q. A combinatorial representation of $(E, \mathcal{B})$ over A is an assignment of a function $f_i : A^r \to A$ to each point $i \in E$ so that

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If $X = \{i_1, \ldots, i_r\}$, we denote $(f_{i_1}, \ldots, f_{i_r})$ by f_X.

An example

Let $n = 4$ and $\mathcal{B} = \{\{1,2\},\{3,4\}\}$. A combinatorial representation over a 3-element set $\{a,b,c\}$ is given by taking f_1 and f_2 to be the two coordinate functions (that is, $f_1(x,y) = x$ and $f_2(x,y) = y$), and f_3 and f_4 by the tables

b	a	a
b	c	b
c	c	a

and

b	b	c
a	c	c
a	b	a

.

Note that $(E,\mathcal{B})$ is not a matroid.

A normalisation

Suppose that $b = \{i_1, \ldots, i_r\} \in \mathcal{B}$. Define functions g_i, for $i \in E$, by

$$g_i(x_1, \ldots, x_r) = f_i(y_1, \ldots, y_r),$$

where $(y_1, \ldots, y_r)$ is the inverse image of $(x_1, \ldots, x_r)$ under the bijection f_b. These functions also define a combinatorial representation, with the property that g_{i_j} is the jth coordinate function. So, where necessary, we may suppose that the first r elements of E form a basis and the first r functions are the coordinate functions. This transformation can be viewed as a change of variables.

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Proof.

We verify the exchange axiom. Let $B_1, B_2 \in \mathcal{B}$; we may assume that the elements of B_1 are the coordinate functions. Now consider the $r-1$ functions f_i for $i \in B_2$, $i \neq k$, for some fixed $k \in B_2$. These define a surjective function from F^r to F^{r-1}. Take any non-zero vector in the kernel, and suppose that its lth coordinate is non-zero. Then it is readily checked that the functions with indices in $B_2 \setminus \{k\} \cup \{l\}$ give a bijection from F^r to F^r; so this set is a basis. □

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Let $(E, \mathcal{B}_1)$ and $(E, \mathcal{B}_2)$ be families of r-sets, which have representations over alphabets of cardinalities q_1 and q_2 respectively. Then $(E, \mathcal{B}_1 \cap \mathcal{B}_2)$ has a representation over an alphabet of size $q_1 q_2$.

Now, to prove the theorem, we observe that

$$\mathcal{B} = \bigcap_{C \notin \mathcal{B}} \left(\binom{E}{r} \setminus \{C\} \right)$$

so it is enough to represent the family consisting of all but one of the r-sets; and it is not too hard to show that this family is indeed a representable matroid.

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Note that our proof shows that in fact every set family has a representation by "matrix functions". More on this later.

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Note that our proof shows that in fact every set family has a representation by "matrix functions". More on this later.

Question

Given a set family, what are the cardinalities of alphabets over which it has a combinatorial representation?

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As a warm-up, let us consider the complete graph. It is readily checked from the definitions that a representation of K_n over an alphabet of size q is the same thing as a set of $n - 2$ mutually orthogonal Latin squares of order q; these are known to exist for all sufficiently large q.

Pairwise balanced designs

A pairwise balanced design, or PBD, consists of a set X and a collection $\mathcal{L}$ of subsets of X (each of size greater than 1) such that every two points of X are contained in a unique "line" in $\mathcal{L}$. If the line sizes all belong to the set K of positive integers, we call it a PBD(K).

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Given K, we define

$$\begin{aligned} \alpha(K) &= \gcd\{k-1 : k \in K\}, \\ \beta(K) &= \gcd\{k(k-1) : k \in K\}. \end{aligned}$$

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This is the essential tool in the proof of our theorem.

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We claim that the set K of alphabet sizes for which the given graph Γ has an idempotent representation is PBD-closed.
Let $(X, \mathcal{L})$ be a PBD, and suppose that Γ has a representation (f^L) with alphabet L, for every line $L \in \mathcal{L}$. Define a representation (f) of Γ over X by the rule that $f_i(x,x) = x$, while if $x \neq y$ then

$$f_i(x,y) = f_i^L(x,y),$$

where L is the unique line containing x and y. It is readily checked that this is a combinatorial representation.

Now it is straightforward to see that the set K of alphabet sizes over which Γ has a combinatorial representation satisfies $\alpha(K) = 1$ and $\beta(K) = 2$. (Using the proof of the first theorem, we see that K contains a sufficiently high power of any prime.)

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Question

Does an analogous result hold for families of r-sets with $r > 2$?

Matrix representations

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We saw that, if two families of sets have representations, then their intersection has a representation given by a "direct product" construction over the Cartesian product of the alphabets.
In particular, if two families have linear representations over F, then their intersection has a "representation by two-rowed matrices", each point associated with a function from $(F^r)^2$ to F^2.

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There are families which do not have representations by two-rowed matrices. An example is given by

$$E = \{1, \ldots, 7\}, \mathcal{B} = \{\{1,2\}, \{3,4\}, \{5,6\}, \{5,7\}, \{6,7\}\}.$$

The proof of non-representability uses the Ingleton inequality.

Rank functions

A rank function for a set family $(E, \mathcal{B})$ is a function $\mathrm{rk} : 2^E \to [0, r]$ satisfying

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The first three conditions are equivalent to the definition of a polymatroid.

Rank functions from representations

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Let $f = (f_i)$ be a representation of $(E, \mathcal{B})$ over an alphabet X of size q. Then the function r_f, defined by $r_f(S) = H(f_S)$, is a rank function for $(E, \mathcal{B})$.

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Here H is the q-ary entropy function given by

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The converse is false; there are rank functions which do not arise from any combinatorial representation.

Bounds for rank functions

If we set $r_m(X) = \max_{B \in \mathcal{B}} |B \cap X|$ and $r_M(X) = \min\{r, |X|\}$, (so that r_M is the rank function for the uniform matroid of rank r), then it is easy to see that $r_m(X) \leq \mathrm{rk}(X) \leq r_M(X)$.

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On the other hand, we have:

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Any family $(E, \mathcal{B})$ has a rank function which takes integer or half-integer values (or indeed, values in the rationals with denominator dividing p, for any $p > 1$).

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An example of such a function is given by

$$\mathrm{rk}(X) = \begin{cases} |X| & \text{if } |X| \leq r-1 \text{ or } X \in \mathcal{B}, \\ r - 1/p & \text{if } |X| = r,\ X \notin \mathcal{B}, \\ r & \text{if } |X| \geq r+1. \end{cases}$$

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We see that the function r_M is the supremum of all rank functions for $(E, \mathcal{B})$, and can be approached arbitrarily closely.

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Let $(E, \mathcal{B})$ be a set family of rank r. Then there is a set X with $|X| = r$ and $r_m(X) = (r + I)/2$, where

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Moreover, for any rank function rk*, we have*

$$\mathrm{rk}(X) - r_m(X) \geq (r - I)/4.$$

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So a basis disjoint from all other bases leads to large differences between any rank function and the lower bound r_m.

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- $\mathrm{rk}(\mathrm{cl}(X)) = \mathrm{rk}(X)$ for all $X \subseteq E$.

Closure

A rank function defines a closure operator cl, by

$$\mathrm{cl}(X) = \{e \in E : \mathrm{rk}(X \cup \{e\}) = \mathrm{rk}(X)\}.$$

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Not every closure operator (satisfying the first three conditions) comes from a rank function.

Closure in a representation

If the rank function arises from a combinatorial representation $f = (f_e : e \in E)$, then we have

$$\mathrm{cl}(X) = \{e \in E : f_X \text{ refines } f_e\}.$$

(We say that f_1 refines f_2 if $f_1(x) = f_1(y)$ implies $f_2(x) = f_2(y)$.)

Network coding

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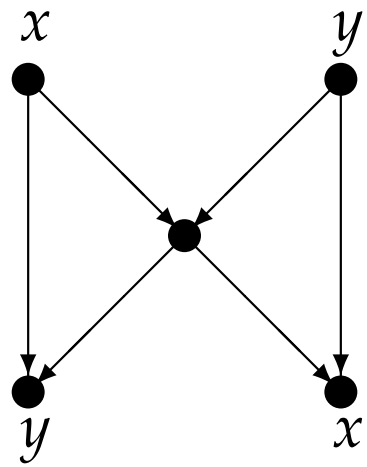
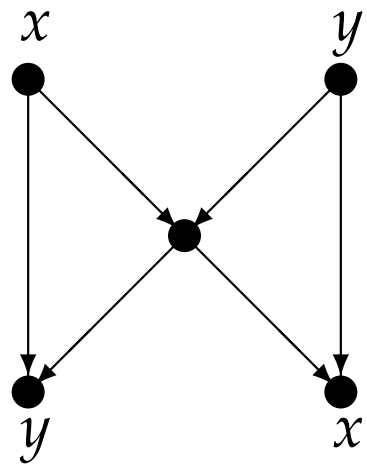
I begin with the butterfly network, the paradigm of the subject.

The butterfly network



The nodes at the top have to send the corresponding messages to the nodes at the bottom.

The butterfly network



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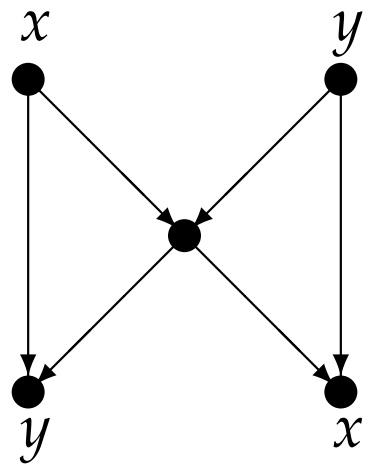

The nodes at the top have to send the corresponding messages to the nodes at the bottom.
In a network carrying ordinary commodities such as water, it would take two time steps because of the bottleneck in the middle.
But with information, the transfer can be done in one time step, if the nodes have the computational capability to do "exclusive or" (addition mod 2) on their inputs.

Guessing number

Given a digraph, in which each vertex holds a symbol from the alphabet A of size q. A vertex is not aware of its own symbol, but knows the symbols of all its out-neighbours.

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We say that the guessing number of the complete directed graph is $n-1$ (the logarithm to base q of the improvement in odds that can be achieved by the best strategy).

A connection

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Suppose that the network has n input nodes required to send characters from an alphabet of size q to n output nodes in specified order. Form a digraph by identifying each input node with the corresponding output node. The guessing number of the resulting digraph is equal to the capacity of the original network.

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So two pairs of nodes of the original network can communicate simultaneously.