# Between primitive and 2-transitive, 1: Synchronization

Peter J. Cameron
(joint work with many other people!)

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# Permutation groups

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Permutations act on the right: that is, the image of $\alpha$ under the permutation $g$ is written $\alpha g$.

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- $G$ is 2-transitive if, for any pairs $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$ of distinct elements of $\Omega$, there exists $g \in G$ with $\alpha_1 g = \beta_1$ and $\alpha_2 g = \beta_2$.

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These conditions get stronger as you go down the list, apart from a small glitch for $n = 2$, where the trivial group is primitive according to the definition. I will only apply the concept of primitivity to transitive groups.

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- $G$ is primitive if it preserves no non-trivial equivalence relation on $\Omega$, or (the same thing) partition of $\Omega$;
- $G$ is 2-transitive if it preserves no non-trivial binary relation on $\Omega$.

# Transitive, primitive, 2-transitive

Any transitive action of a permutation group is isomorphic to the action on the set of right cosets of a subgroup by right multiplication. (Take the subgroup to be the stabiliser $\mathrm{Stab}_G(\alpha)$ of a point $\alpha$: for each $\beta$, the set of group elements mapping $\alpha$ to $\beta$ is a right coset of $Stab_G(\alpha)$.)

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Now an action is primitive if and only if $\mathrm{Stab}_G(\alpha)$ is a maximal (proper) subgroup of $G$.

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The situation for primitive groups is less satisfactory; I turn to this now.

# O'Nan–Scott

The study of the socle (the product of the minimal normal subgroups) of a primitive group goes back to Jordan, who proved most of what we now know as the O'Nan–Scott theorem (or Aschbacher–O'Nan–Scott theorem). This theorem was stated independently by O'Nan and Scott in the preliminary proceedings of the Santa Cruz Conference on Finite Groups in 1979, though only Scott's version made it into the final Proceedings. A small error in the statement was corrected by Aschbacher.

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My formulation of it is a little different from most of the standard ones, but I think it is perhaps the simplest.

# Basic and non-basic groups

A power structure, or Cartesian structure, on $\Omega$ is an identification of $\Omega$ with the set $A^l$ of all $l$-tuples over an alphabet $A$ of size $k$. This implies that $n = k^l$.

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Now we say that a primitive permutation group $G$ is basic if it preserves no non-trivial power structure on $\Omega$.

# Non-basic groups

A non-basic group is contained in a wreath product $S_k \operatorname{Wr} S_l$ of symmetric groups; indeed, it is contained in $K \operatorname{Wr} L$, where the appropriate $K$ and $L$ can be extracted from $G$. (For example, $L$ is the permutation group induced by $G$ on the set of coordinates of the power structure.) Moreover, we can assume that $K$ is primitive and $L$ is transitive.

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The first part of the O'Nan–Scott theorem asserts that the socle of a non-basic primitive group $G$ is $N^l$, where $N$ is either the socle of $K$ or a minimal normal subgroup of $K$. (The latter case is only relevant when the socle of $K$ is the product of two minimal normal subgroups; this is the twisted wreath product case pointed out by Aschbacher.

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The smallest twisted wreath product has degree $n = 60^6$, too big for computation at the moment.

# Basic groups

We introduce three types of primitive permutation groups:

- Affine groups, whose minimal normal subgroup is elementary abelian. Such a group is the semidirect product of the additive group of a vector space $V$ with a subgroup $H$ of the general linear group on $V$. Now $G$ is primitive if and only if $H$ is irreducible, and $G$ is basic if and only if $H$ is a primitive linear group.

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- Almost simple groups, satisfying $T \leq G \leq \mathrm{Aut}(T)$, where $T$ is simple (but the action is unspecified).

Now the second part of the O'Nan–Scott Theorem asserts that a basic primitive group is of one of these three types.

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João Araújo's intuition is that the synchronizing property is closely related to primitivity. As we will see, this is true, but things are not quite so simple.

# A characterisation

**Theorem**

*The permutation group $G$ on $\Omega$ fails to be synchronizing if and only if there is a non-trivial partition $\pi$ of $\Omega$ and a subset $S$ of $\Omega$ such that $Sg$ is a section (or transversal) for $\pi$ for all $g \in G$.*

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**Proof.**

If $\pi$ and $S$ as in the theorem exist, let $f$ map each point of $\Omega$ to the representative in $S$ of its $\pi$-class. Conversely, if $G$ is not synchronizing, let $f$ be an element of minimal rank such that $\langle G, f \rangle$ contains no element of rank 1, and let $\pi$ and $S$ be the kernel and image of $f$. $\square$

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Of course it is equivalent to say that $S$ is a section for $\pi g$ for all $g \in G$.

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If $G$ preserves a power structure $A^l$, take $\pi$ to be the partition of $A^l$ according to the entry in the first coordinate, and $S$ the diagonal set $\{(x, x, \ldots, x) : x \in A\}$. $\square$

# What is synchronizing?

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The clique number $\omega(X)$ of $G$ is the size of the largest complete subgraph of $X$, and the chromatic number $\chi(X)$ is the smallest number of colours required for a proper vertex-colouring of $X$ (one in which adjacent vertices get different colours).

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Clearly $\omega(X) \leq \chi(X)$, since the vertices of a clique must all get different colours in a proper colouring.

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Conversely, if the partition $\pi$ and section $S$ demonstrate that $G$ is non-synchronizing, form a graph on $\Omega$ by the rule that $\alpha$ and $\beta$ are adjacent if and only if some $g \in G$ maps them both into $S$. $\square$

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The theorem also gives a simple proof that synchronizing groups are primitive. If $G$ is imprimitive, it preserves a disjoint union of $l$ complete graphs of size $k$ (where $kl = n$), which has clique number and chromatic number $k$.

## An example

Let $G$ be the symmetric group $S_m$ acting on the set $\Omega$ of 2-element subsets of $\{1, \ldots, m\}$, with $n = \binom{m}{2}$. Then $G$ is primitive (and basic) for $m \geq 5$.

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The line graph of $K_m$ has clique number $m - 1$ (all pairs containing a fixed point) and chromatic number $m - 1$ if $m$ is even, $m$ if $m$ is odd (the tournament-scheduling problem). So $G$ is non-synchronizing if $m$ is even.

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The complement has clique number $\lfloor m/2 \rfloor$ and chromatic number $m - 2$; these are never equal for $m \geq 5$. So $G$ is synchronizing if $m \geq 5$ and $m$ is odd.

# Non-synchronizing ranks

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One approach is via the notion of non-synchronizing ranks. We say that the number $k < n$ is a *non-synchronizing rank* for the permutation group $G$ on $\Omega$ if there is a map $f : \Omega \to \Omega$ of rank $k$ such that $\langle G, f \rangle$ does not contain a transformation of rank 1. Let $\mathrm{NS}(G)$ be the set of all non-synchronizing ranks for $G$.

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Our hypothesis is that

- $\mathrm{NS}(G)$ is large if $G$ is imprimitive;
- $\mathrm{NS}(G)$ is (very) small if $G$ is primitive.

# Imprimitive groups

### Theorem

*There is a constant $c > 0$ such that, if $G$ is imprimitive of degree $n$, then $|\mathrm{NS}(G)| \geq cn$. In fact, the lim inf of $|\mathrm{NS}(G)|/n$ for imprimitive groups $G$ is $\frac{3}{4}$.*

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**Proof.**

Suppose that $G$ has an invariant partition with $l$ blocks of size $k$. Then $\langle G, f \rangle$ has no rank 1 elements for the following maps $f$:

- $f$ maps each block bijectively to a block, but maps two blocks to the same place (rank $kx$ for $1 \leq x \leq l - 1$);
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The numerical tailpiece is an exercise. $\square$

# Primitive groups

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Certain non-basic groups realise powers of $\log n$, but we don't even know that we have counted all their non-synchronizing ranks.

# Another measure

Another measure of how non-synchronizing a group is, was introduced by João Araújo and Wolfram Bentz. Let the partition $\pi$ and section $S$ demonstrate that $G$ is non-synchronizing. Consider the graph $X$ in which two points of $S$ are joined if no element of $G$ maps both into $S$. Then each part $T$ of $S$ is a clique in this graph. Let $k$ be the smallest number such that the intersection of the closed neighbourhoods of any $k$ points of $T$ is precisely $T$.

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Then $k = 1$ if and only if $T$ is a block of imprimitivity for $G$.
They have some strong results on the case $k = 2$, but I have not time to discuss them here.

# A counting argument

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Let $A$ and $B$ be subsets of $\Omega$, with $|A| = a$ and $|B| = b$. Now an easy counting argument shows that the average cardinality of $Ag \cap B$, as $g$ runs over $G$, is $ab/n$. (Count triples $(x, y, g)$ with $x \in A$, $y \in B$ and $g \in G$ with $xg = y$. There are $a$ choices of $x$, $b$ choices of $y$, and $|G|/n$ choices of $g$. Counted otherwise, for each $g \in G$, $ag \in Ag \cap B$, so there are $|Ag \cap B|$ choices of $a$ and $b$.

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Hence one of two possibilities holds:

- $|Ag \cap B| = ab/n$ for all $g \in G$ (and $n$ divides $ab$);
- there exist $g_1, g_2 \in G$ with $|Ag_1 \cap B| > ab/n$ and $|Ag_2 \cap B| < ab/n$.

# A consequence for synchronization

If $G$ is transitive on $\Omega$, $\pi$ a partition of $\Omega$ and $S$ a subset of $\Omega$ such that $Sg$ is a section for $\pi$ for all $g \in G$, then $\pi$ is uniform (all parts have the same size). For each part meets $Sg$ in precisely one point, for all $g \in G$.

# Related properties

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- $G$ has Property X if there do not exist non-trivial partitions $\pi_1$ and $\pi_2$ of $\Omega$ such that, for every $g \in G$, each part of $\pi_2 g$ is a section for $\pi_1$.

The silly name for the last property is because nobody has thought much about it. Perhaps it could be called "gridless"?

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- All properties mentioned so far (transitive, primitive, basic, property X, synchronizing, separating, 2-transitive) are closed upwards.
- None of the implications above reverses.

# Graph-theoretic interpretations

Both separation and Property X have graph-theoretic interpretations. Let $\overline{X}$ denote the complement of $X$, the graph whose edges are the non-edges of $X$.

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### Theorem

- *A transitive permutation group $G$ on $\Omega$ is separating if and only if there is no non-trivial graph $X$ on $\Omega$ such that* $\omega(X) \cdot \omega(\overline{X}) = |\Omega|$.

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The proof is similar to the proof of the analogous statement for synchronization.

# The implications don't reverse

We have seen a basic group which is not synchronizing. Indeed, it is easy to find a basic group which doesn't have Property X, and our earlier example of a non-synchronizing group (the symmetric group $S_m$ acting on 2-sets for $m$ even, $m > 5$) has Property X.

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Most synchronizing groups are separating, but many are not 2-transitive.

## Graph homomorphisms

Our graph-theoretic interpretation of the properties connects up with the active area of graph homomorphisms. A homomorphism from a graph $X$ to a graph $Y$ is a map from the vertices of $X$ to the vertices of $Y$ which maps edges to edges. We don't specify what happens to a non-edge, which may map to a non-edge or to an edge or even collapse to a single vertex.

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So, if $X$ is a non-trivial graph with $\omega(X) = \chi(X)$, then the endomorphism monoid of $X$ is a transformation monoid containing the automorphism group of $G$ and no rank 1 transformation.

# Further up the hierarchy

What about transitive groups with the following property:

> *There do not exist non-trivial subsets $A, B$ of $\Omega$ such that $|Ag \cap B| = |A| \cdot |B|/n$ for all $g \in G$.*

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Some variants of this property are related to the synchronization project.

# $\mathbb{Q}I$-groups

If we replace the sets $A, B$ in the previous property by multisets, and the set intersection is defined so that an element $x$ with multiplicity $m_A(x)$ in $A$ and $m_B(x)$ in $B$ contributes $m_A(x)m_B(x)$ to the intersection, then we obtain the so-called Property $\mathbb{Q}I$, which states that the only $\mathbb{Q}G$-submodules of the permutation module $\mathbb{Q}\Omega$ are those invariant under the symmetric group.

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The $\mathbb{Q}I$-groups were recently determined by Bamberg, Giudici, Liebeck, Praeger and Saxl.
One can make this definition over any field $\mathbb{F}$. Can anything interesting be said about $\mathbb{F}I$ groups? (For example, when $\mathbb{F}$ is a finite field?)