Sudoku, Mathematics and Statistics

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Sudoku

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Instructions in *The Independent*

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But who invented Sudoku?

- Leonhard Euler
- W. U. Behrens
- John Nelder
- Howard Garns
- Robert Connelly

Euler

Euler posed the following question in 1782.

> *Of 36 officers, one holds each combination of six ranks and six regiments. Can they be arranged in a* 6×6 *square on a parade ground, so that each rank and each regiment is represented once in each row and once in each column?*

NO!!

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Example

$\circ$	a	b	c
a	b	a	c
b	a	c	b
c	c	b	a

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- There is a Markov chain method to choose a random Latin square. But we don't know much about what a random Latin square looks like.
- For example, the second row is a permutation of the first; this permutation is a derangement (i.e. has no fixed points). Are all derangements roughly equally likely?

Orthogonal Latin squares

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But we don't know

- how many orthogonal pairs of Latin squares of order n there are;
- the maximum number of mutually orthogonal Latin squares of order n;
- how to choose at random an orthogonal pair.

Latin squares in statistics

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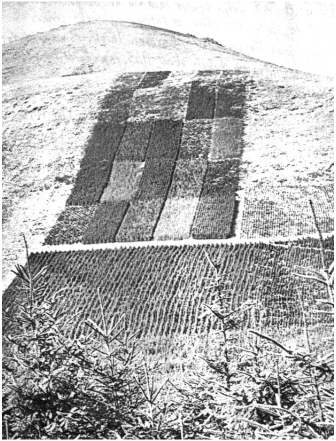

A Latin square in Beddgelert Forest, designed by R. A. Fisher.

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Example

Suppose that there is a boggy patch in the middle of the field.

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

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How difficult is it to recognise a critical set, or to complete one?

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Garns called his puzzle "number place". It became popular in Japan under the name "Sudoku" in 1986 and returned to the West a couple of years ago.

Connelly

Robert Connelly proposed a variant which he called symmetric Sudoku. The solution must be a gerechte design for all these regions:

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
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Subsquares

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6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
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Broken rows

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Locations

Coordinates

We coordinatise the cells of the grid with F^4, where F is the integers mod 3, as follows:

- the first coordinate labels large rows;
- the second coordinate labels small rows within large rows;
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Now Connelly's regions are cosets of the following subspaces:

Rows	$x_1 = x_2 = 0$	Columns	$x_3 = x_4 = 0$
Subsquares	$x_1 = x_3 = 0$	Broken rows	$x_2 = x_3 = 0$
Broken columns	$x_1 = x_4 = 0$	Locations	$x_2 = x_4 = 0$

Affine spaces and SET®

The card game SET has 81 cards, each of which has four attributes taking three possible values (number of symbols, shape, colour, and shading). A winning combination is a set of three cards on which either the attributes are all the same, or they are all different.

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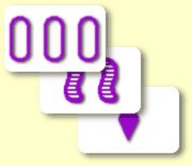

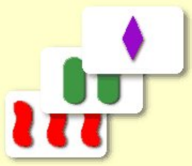

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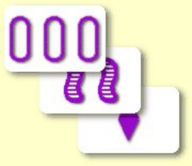

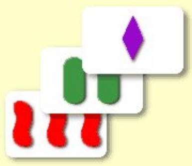

Each card has four coordinates taken from F (the integers mod 3), so the set of cards is identified with the 4-dimensional affine space. Then the winning combinations are precisely the affine lines!

Perfect codes

A code is a set C of "words" or n-tuples over a fixed alphabet F. The Hamming distance between two words v, w is the number of coordinates where they differ; that is, the number of errors needed to change the transmitted word v into the received word w.

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A code C is e-error-correcting if there is *at most* one word at distance e or less from any codeword. [Equivalently, any two codewords have distance at least $2e + 1$.] We say that C is perfect e-error-correcting if "at most" is replaced here by "exactly".

Perfect codes and symmetric Sudoku

Take a solution to a symmetric Sudoku puzzle, and look at the set S of positions of a particular symbol s. The coordinates of the points of S have the property that any two differ in at least three places; that is, they have Hamming distance at least 3. [For, if two of these words agreed in the positions 1 and 2, then s would occur twice in a row; and similarly for the other pairs.]

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So a symmetric Sudoku solution is a partition of F^4 into nine perfect codes.

All symmetric Sudoku solutions

Now it can be shown that a perfect code C in F^4 is an affine plane, that is, a coset of a 2-dimensional subspace of F^4. To show this, we use the SET® principle: We show that if $v, w \in C$, then the word which agrees with v and w in the positions where they agree and differs from them in the positions where they differ is again in C.

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It is not hard to show that there are just two ways to do this.

One solution consists of nine cosets of a fixed subspace.

There is just one further type, consisting of six cosets of one subspace and three of another. [Take a solution of the first type, and replace three affine planes in a 3-space with a different set of three affine planes.]

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An earlier computation by Felgenhauer and Jarvis gives the total number of solutions to be 6 670 903 752 021 072 936 960. Now for each conjugacy class of non-trivial symmetries of the grid, it is somewhat easier to calculate the number of fixed solutions.

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The last two questions are particularly interesting in the case where $n = kl$ and the regions are $k \times l$ rectangles.

References

- R. A. Bailey, P. J. Cameron and R. Connelly, Sudoku, Sudoku, gerechte designs, resolutions, affine space, spreads, reguli, and Hamming codes, *American Math. Monthly*, to appear. Preprint available from `http://www.maths.qmul.ac.uk/~pjc/preprints/sudoku.pdf`