Multi-letter Youden rectangles from quadratic forms

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In memory of J. J. Seidel

History

I am completing a story which began (for me) with the paper


(This paper, and in particular the object shown on the next slide, grew from a conversation on the bus, on the conference excursion at the British Combinatorial Conference in Aberystwyth, 1973.)

A design

<table>
<thead>
<tr>
<th>AAA</th>
<th>BBB</th>
<th>CCC</th>
<th>DDD</th>
<th>EEE</th>
<th>FFF</th>
<th>GGG</th>
<th>HHH</th>
</tr>
</thead>
<tbody>
<tr>
<td>BCC</td>
<td>DDD</td>
<td>EEE</td>
<td>FFF</td>
<td>GGG</td>
<td>HHH</td>
<td>IJJ</td>
<td>KKK</td>
</tr>
<tr>
<td>LLL</td>
<td>MMM</td>
<td>NNN</td>
<td>OOO</td>
<td>PPP</td>
<td>QQQ</td>
<td>RRR</td>
<td>SSS</td>
</tr>
</tbody>
</table>

What is it?

- Three MOLS.
- In the first six rows, each Latin rectangle is a symmetric $(16,6,2)$ BIBD; the blocks of one design meet the blocks of another in 1 or 3 points; and if we let a column of one rectangle be incident with a column of another if they share one letter, then again we get a symmetric $(16,6,2)$ BIBD. Moreover, these three BIBDs are “linked” in the same way.
- Similarly for the last ten rows.
Viewed otherwise

More symmetrically, we have:

(a) Four sets $S_1, \ldots, S_4$ each of size 16, with an incidence relation between each pair, so that $(S_i, S_j)$ forms a $(16, 6, 2)$ BIBD, and the number of elements of $S_k$ incident with $s_i$ and $s_j$ is one if $s_i$ and $s_j$ are incident, 3 otherwise (this is the linking condition);

(b) a collection of 96 sets each consisting of four pairwise incident points, (with each incident pair lying in exactly one such set), partitioned into 6 sets of size 16, so that each point occurs once in each part. These 4-sets are cliques in the incidence graph; so we have a resolvable clique-cover.

So what’s new?

Condition (a) defines a system of linked square designs or SLSD. Infinite families of examples were found in my first joint paper:


This paper gives two families, one linear, one non-linear (related to the Kerdock codes).

It has taken me more than 25 years to realise that, for the linear (but not for the non-linear) examples, condition (b) also holds. Thus we obtain an infinite family of objects generalising the one given earlier.

To prove this, one just has to use some easy facts about the automorphism group of the SLSDs.

Or again . . .

From a statistician’s point of view, the 96 cliques are the plots or experimental units. We have four partitions $\pi_1, \ldots, \pi_4$ of the set $\pi$ of plots into 16 sets of 6, and one partition $\sigma$ into 6 sets of 16, satisfying:

- $\pi_i$ and $\sigma$ are orthogonal for all $i$.
- $\pi_i$ and $\pi_j$ form a BIBD for all $i \neq j$.
- $\pi_i$, $\pi_j$, and $\pi_k$ satisfy the “linking condition” for all distinct $i, j, k$.

The rectangle given before is obtained by taking $\sigma$ as “rows” and $\pi_1$ as “columns”; the other three partitions give the three “letters” in the array.

Details

The BIBDs have parameters $(2^{2m}, 2^{2m-1} \pm 2^{m-1}, 2^{2m-2} \pm 2^{m-1})$ for any $m \geq 2$. The number of sets in the SLSD is $2^m$.

To construct them, we identify $GF(2^{2m})^2$ with $GF(2)^{2m}$ by restricting scalars. We take $S_\lambda$ to be the set of quadratic forms which polarise to the bilinear form $\text{Trace}(\lambda b)$ for all $\lambda \in GF(2^m)$, where $b((x_1, y_1), (x_2, y_2)) = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$. Then for $\lambda \neq \mu$, if $Q_\lambda \in S_\lambda$ and $Q_\mu \in S_\mu$, then $Q_\lambda - Q_\mu$ is non-degenerate; we declare $Q_\lambda$ and $Q_\mu$ incident if $Q_\lambda - Q_\mu$ has type $\epsilon$, where $\epsilon = \pm 1$.

The group $2^{2m} : \text{SL}(2, 2^m)$ acts on the configuration. The stabiliser of an incident pair is a dihedral group fixing one element of each set $S_\lambda$; using the group, we see that these elements are mutually incident, so they form a clique. Thus we have our clique-cover. Now we partition them into orbits of the translation group to get the resolution.