The profile of a relational structure

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The profile is the sequence \((f_0,f_1,f_2,\ldots)\), were \(f_n\) is the number of \(n\)-element structures in the age, up to isomorphism.
Examples

- An infinite linear order
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  - Age: all finite linear orders

- A disjoint union of edges
  - Age: All finite unions of edges and isolated vertices
  - Profile: $f_n = \lfloor n/2 \rfloor + 1$

- An infinite path
  - Age: All finite unions of paths
  - Profile: $f_n = p(n)$ (partitions of $n$)

- A totally ordered set coloured with $k$ colours, each colour class dense
  - Age: words in an alphabet of size $k$
  - Profile: $f_n = k^n$
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A universal graph
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  ▶ Profile: \( f_n \sim \frac{2^{n(n-1)/2}}{n!} \)
Let $G$ be a permutation group on the countably infinite set $\Omega$. Then there is a relational structure $R$ on $\Omega$ such that

- $G$ is contained in the automorphism group of $\Omega$;
- if two finite substructures of $R$ are isomorphic, then there is an element of $G$ inducing the given isomorphism between them.

This means that $R$ is homogeneous, and that $G$ is a dense subgroup of its automorphism group (in the topology of pointwise convergence).

So the profile of $R$ also counts orbits of $G$ on $n$-element subsets of $\Omega$ for $n = 0, 1, 2, \ldots$. 
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Permutation groups

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So the profile of $R$ also counts orbits of $G$ on $n$-element subsets of $\Omega$ for $n = 0, 1, 2, \ldots$. 
Quite a lot is known globally about the growth of a profile:

- Either an and $f_n \leq bn$ for some natural number $d$ and $a$, $b > 0$; or $f_n$ grows faster than a polynomial in $n$.
- In the latter case, $f_n \geq \exp(n^{1/2} - \epsilon)$ for sufficiently large $n$.
- (These two results assume that the number of relations is finite).

- In the case of a primitive permutation group (one preserving no non-trivial equivalence relation), there is a constant $c > 1$ such that either $f_n = 1$ for all $n$, or $f_n \geq c^{n/p(n)}$ for some polynomial $p$.
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▶ In the case of a primitive permutation group (one preserving no non-trivial equivalence relation), there is a constant \( c > 1 \) such that either \( f_n = 1 \) for all \( n \), or \( f_n \geq c^n / p(n) \) for some polynomial \( p \).
Local conditions

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**Theorem**

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There are two known proofs of this theorem; one using a Ramsey-type theorem (outlined on the next slide), the other using finite combinatorics and linear algebra (see later).
A Ramsey-type theorem

Given a colouring of the \( n \)-sets with colours \( c_1, \ldots, c_r \), we say that the colour scheme of an \((n + 1)\)-set \( S \) is the \( r \)-tuple \((a_1, \ldots, a_r)\), where \( a_i \) is the number of sets of colour \( c_i \) in \( S \).
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Theorem

Let the $n$-subsets of an infinite (or sufficiently large finite) set $\Omega$ be coloured with $r$ colours (all of which are used). Then there are at least $r$ colour schemes of $(n + 1)$-sets. In fact, there exist $(n + 1)$-sets $T_1, \ldots, T_r$ so that $T_i$ contains a set of colour $c_i$ but none of colour $c_j$ for $j > i$. The "Ramsey numbers" associated with this theorem are not known.
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The age algebra

Let $V_n$ be the complex vector space of all functions from $\binom{\Omega}{n}$ to $\mathbb{C}$ which are constant on isomorphism classes (or $G$-orbits). Thus, $\dim(V_n) = f_n$. 
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There is a multiplication defined on $A = \bigoplus_{n \geq 0} V_n$ as follows: for $f \in V_n$, $g \in V_m$, and $X \in \binom{\Omega}{m+n}$, put

$$(fg)(X) = \sum_{Y \in \binom{X}{n}} f(Y)g(X \setminus Y).$$

The multiplication is commutative and associative, and the constant function $1 \in V_0$ is the identity. So $A$ is a graded algebra with Hilbert series $\sum f_n x^n$.

In the fourth of our examples, $A$ is the shuffle algebra on $k$ symbols.
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**Theorem**

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This theorem is proved by finite combinatorial arguments. It implies that multiplication by $e$ is a monomorphism from $V_n$ to $V_{n+1}$, and hence

$$f_n = \dim(V_n) \leq \dim(V_{n+1}) = f_{n+1}$$

for any $n$. 

Two conjectures

A relational structure $R$ is said to be **inexhaustible** if there is no point whose removal makes the age strictly smaller. In the group case, this holds if and only if $G$ has no finite orbits.

Some time ago I conjectured the group case of the following.

**Conjecture**

Assume that $R$ is inexhaustible. Then

- $A$ is an integral domain (that is, has no zero-divisors);
- $e$ is prime in $A$ (that is, $A/\langle e \rangle$ is an integral domain).

The first of these conjectures has very recently been proved by Maurice Pouzet.
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Assume that $R$ is inexhaustible. Then $f_{m+n} \geq f_m + f_n - 1$.

In outline: multiplication induces a map from the *Segre variety* (the rank 1 tensors modulo scalars) in $V_m \otimes V_n$ into $V_{m+n}$ modulo scalars; so the dimension of $V_{m+n}$ is at least as great as that of the Segre variety.
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In a similar way, if the second part of the conjecture is true, then the profile of an inexhaustible structure would satisfy $g_{m+n} \geq g_m + g_n - 1$, where $g_n = f_{n+1} - f_n$. (Apply a similar argument to $A/\langle e \rangle$, whose $n$th homogeneous component is $V_{n+1}/eV_n$, with dimension $f_{n+1} - f_n$.)
Sketch proof

Let $\Omega$ be a set, $\mathbb{K}$ a field with characteristic zero. Let $f : \binom{\Omega}{n} \to \mathbb{K}$. The support of $f$ is $\{X \in \binom{\Omega}{n} : f(X) \neq 0\}$. A set $T$ is a transversal to a family $\mathcal{H}$ of sets if $T \cap H \neq \emptyset$ for all $H \in \mathcal{H}$. The transversality of $\mathcal{H}$ is the cardinality of the smallest transversal.

Pouzet proved:

Theorem

Given $m, n \geq 0$, there exists $t$ such that, for any $\Omega$ with $|\Omega| \geq m + n$, any field $\mathbb{K}$ of characteristic zero, and any two non-zero maps $f : \binom{\Omega}{n} \to \mathbb{K}$, $g : \binom{\Omega}{m} \to \mathbb{K}$ such that $fg = 0$, the transversality of $\text{supp}(f) \cup \text{supp}(g)$ is at most $t$.

The result follows since removal of a transversal would decrease the age, which is impossible in an inexhaustible structure.
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The result follows since removal of a transversal would decrease the age, which is impossible in an inexhaustible structure.
The theorem is a Ramsey-type theorem, and one can ask for an evaluation of $\tau(m, n)$, the smallest number $t$ for which the conclusion of the theorem is true. It is not hard to show that $\tau(1, n) = 2n$: this is the combinatorics underlying the proof that $f_n \leq f_{n+1}$.
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Pouzet’s proof shows that

$$7 \leq \tau(2, 2) \leq 2(R_k^2(4) + 2),$$

where $k = 5^{30}$ and $R_k^2(4)$ is the classical Ramsey number, the least $p$ such that in any $k$-colouring of the edges of the complete graph on $p$ vertices, there is a monochromatic subgraph of order 4.
Ramsey numbers

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This is rather a large gap – can it be reduced?
Where next?

The conjecture that, if $R$ is inexhaustible, then $e$ is prime in $A(R)$, remains to be proved.
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A more interesting possibility involves showing that, under suitable hypotheses to be determined, if $f_1, \ldots, f_r \in V_n$ and $g_1, \ldots, g_r \in V_m$ are linearly independent, then

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If this were true, the dimension argument would give a much stronger lower bound for \( f_{m+n} \) in terms of \( f_m \) and \( f_n \).
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If this were true, the dimension argument would give a much stronger lower bound for $f_{m+n}$ in terms of $f_m$ and $f_n$.

But it cannot be true in general since the earlier bound is tight in some cases!