Optimal designs and root systems

Peter J. Cameron

Queen Mary
University of London

p.j.cameron@qmul.ac.uk

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Block designs

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The **incidence matrix** $N$ of the block design is the $v \times b$ matrix with $(p, b)$ entry 1 if $p \in B$, 0 otherwise. The matrix $\Lambda = NN^\top$ is the **concurrence matrix**, with $(p, q)$ entry equal to the number of blocks containing $p$ and $q$. It is symmetric, with row and column sums $rk$, and diagonal entries $r$. 
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A 2-design is optimal in all three senses. But what if no 2-design exists for the given $v, k, r$?
For a 2-design, the concurrence matrix is $\Lambda = (r - \lambda)I + \lambda J$, where $J$ is the all-1 matrix. Ching-Shui Cheng suggested looking for designs where $\Lambda$ is a small perturbation of this, say $\Lambda = (r - t)I + tJ - A$, where $A$ is a matrix with small entries (say 0, +1, −1). For E-optimality, we want $A$ to have smallest eigenvalue as large as possible (say greater than −2).
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So we want a square matrix \( A \) such that

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Call such a matrix admissible.
If $A$ is admissible, then $2I + A$ is positive definite, so is a matrix of inner products of a set of vectors in $\mathbb{R}^n$. 

(This idea was originally used by Cameron, Goethals, Seidel and Shult in 1979 for graphs with least eigenvalue $\geq -2$.) So we try to determine the admissible matrices by looking for subsets of the root systems.
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These vectors form a subsystem of a root system of type $A_n, D_n, E_6, E_7$ or $E_8$ (as in the classification of simple Lie algebras by Cartan and Killing). Indeed, they form a basis for the root system.
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The case $A_n$

The vectors of $A_n$ are of the form $e_i - e_j$ for $1 \leq i, j \leq n + 1, i \neq j$, where $e_1, \ldots, e_{n+1}$ form a basis for $\mathbb{R}^{n+1}$. 
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So an admissible matrix of this type is represented by a tree with oriented edges. (We have an edge $j \rightarrow i$ if $e_i - e_j$ is in our subset.)
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An oriented tree gives an admissible matrix if and only if $s(w) - s(v) = c + 2$ for any edge $v \to w$, where $s(v)$ is the signed degree (number of edges in minus number out) and $c$ is the constant row sum.
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Here is an example (edges directed upwards).
The case $D_n$

The vectors of $D_n$ are those of the form $\pm e_i \pm e_j$ for $1 \leq i < j \leq n$, where $e_1, \ldots, e_n$ form an orthonormal basis for $\mathbb{R}^n$. 
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This case is a bit more complicated. An admissible matrix is represented by a unicyclic graph, whose edges are either directed (if of form $e_i - e_j$) or undirected and signed (if of the form $\pm(e_i + e_j)$). A similar condition for constant row sum can be formulated.
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Here is an example:
The case $E_n$

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Here is an example in $E_8$:

$$
\begin{bmatrix}
0 & - & + & + & - & - & + & - \\
- & 0 & - & - & + & + & - & + \\
+ & - & 0 & + & - & - & 0 & 0 \\
+ & - & + & 0 & - & - & 0 & 0 \\
- & + & - & - & 0 & + & 0 & 0 \\
- & + & - & - & + & 0 & 0 & 0 \\
+ & - & 0 & 0 & 0 & 0 & 0 & - \\
- & + & 0 & 0 & 0 & 0 & - & 0 \\
\end{bmatrix}
$$
Having determined the matrices, we can use Leonard Soicher’s DESIGN software to look for block designs. Many examples exist.
Conclusion

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An example in $E_6$ has point set $\{1, 2, 3, 4, 5, 6\}$ and blocks

$\{123, 125, 125, 134, 136, 136, 146, 156, 234, 245, 246, 246, 256, 345, 345, 356\}$. 
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$$\{123, 125, 125, 134, 136, 136, 146, 156, 234, 245, 246, 246, 256, 345, 345, 356\}.$$ 

The next step would be to go on and decide whether any $E$-optimal block designs are obtained in this way. This has not yet been done!