

# Bases for structures and permutation groups

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Is this really necessary?

# Bases

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For example, a basis in a vector space has the property that different vectors have different expressions as linear combinations of basis vectors.

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We'd like to have an efficient method to recognise this difference. This would have practical implications for graph isomorphism, as well as its theoretical interest.

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A set  $S$  of vertices is a **determining set** for a graph if different points outside  $S$  have different neighbour sets in  $S$ . The **determining number** is the size of the smallest such set.

## Base size for permutation groups

If the permutation group  $G$  on  $\Omega$  (where  $|\Omega| = n$ ) has a base of size  $b$ , then  $|G| \leq n^b$ . Moreover, if  $B$  is a base of minimum size, then  $|G| \geq 2^b$ .

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Clearly any determining set for a graph is a base for its automorphism group; so the minimal base size does not exceed the determining number.

## Babai's Theorem

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### Theorem

*Let  $G$  be a primitive but not 2-transitive permutation group of degree  $n$ . Then, for any pair  $x, y$  of distinct points, there are at least  $(\sqrt{n} - 1)/2$  points  $z$  for which  $(x, z)$  and  $(y, z)$  lie in different  $G$ -orbits.*

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This is proved by a detailed analysis of the coherent configuration associated with  $G$ . (I say more about coherent configurations later.)

## Babai's Theorem, 2

Now consider the hypergraph whose vertices are the pairs of points of  $\Omega$ , and whose edges are indexed by points of  $\Omega$ ; the edge labelled  $z$  consists of the pairs “distinguished” by  $z$ . A theorem of Lovász shows that there are  $b = 4\sqrt{n} \log n$  edges which cover all vertices – these edges can be chosen at random and cover with non-zero probability.

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The bound is best possible up to a factor of  $c \log n$  in the exponent.

## The $\log n$ factor

In situations like this we expect a  $\log n$  factor. The simplest possible example is the following:

### Theorem

*Suppose that a  $k$ -uniform hypergraph on  $n$  points has a vertex-transitive automorphism group. Then there is a set of at most  $(n/k) \log n$  edges that cover the vertex set.*

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Choose  $m$  images of a fixed edge under random automorphisms. The probability of a given vertex being uncovered is  $(1 - k/n)^m$ , and so the expected number of uncovered vertices is  $n(1 - k/n)^m$ . If this is less than 1, then there is a choice with no uncovered vertices.

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Can we get rid of it by more intricate combinatorics?

## Symmetry and logic, 1

One of the most remarkable theorems about symmetry for finite and countably infinite structures was proved by Engeler, Ryll-Nardzewski and Svenonius in 1959. Structures here are allowed to have relations (graphs, orders, hypergraphs) and functions (groups, rings). All structures are (at most) countable.

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Cantor's theorem shows that  $(\mathbb{Q}, <)$  is countably categorical (it is the unique countable dense total order without endpoints).

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The group of order-preserving permutations of  $\mathbb{Q}$  is oligomorphic: two  $n$ -tuples of distinct elements lie in the same orbit if and only if they are themselves order-isomorphic – we can extend the order-isomorphism to a piecewise-linear map on  $\mathbb{Q}$  – so there are  $n!$  orbits on  $n$ -tuples of distinct elements.

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More is true. If  $M$  is countably categorical, then two  $n$ -tuples lie in the same orbit of  $\text{Aut}(M)$  if and only if they satisfy the same first-order formulae (that is, they have the same **first-order type**).

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So our philosophical principle holds for first-order structure.

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A graph is said to be  **$t$ -strongly regular** if, for any set  $S$  of vertices with  $|S| \leq t$ , the number of common neighbours of  $S$  depends only on the isomorphism type of the induced subgraph on  $S$ .

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In these cases, almost all such structures have no non-trivial automorphisms.

## Digression

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- ▶ A theorem of Neumaier shows that strongly regular graphs with least eigenvalue  $-m$  (an integer) are complete multipartite with parts of size  $m$ , or line graphs of linear spaces or transversal designs with block size  $m$ , or one of a finite list  $\mathcal{L}(m)$  of exceptions. Laci Babai showed that almost all Steiner triple systems (linear spaces with block size 3) have trivial automorphism group; the same is true for Latin squares (equivalent to transversal designs with block size 3).

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- ▶ There are known to be 32548 strongly regular graphs with parameters  $(36, 15, 6, 6)$ ; all but 11 of them belong to the list  $\mathcal{L}(3)$ . Most have trivial automorphism group (but I don't have the exact number).

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A graph is  *$t$ -homogeneous* if any isomorphism between sets of at most  $t$  vertices can be extended to an automorphism of the graph. Clearly  $t$ -homogeneity implies  $t$ -strong regularity.

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This and some related results suggest that perhaps there is an absolute bound on the number of variables required in formulae “labelling” the vertices of a graph in terms of a base for its automorphism group.

## A test case: Paley graphs

Let  $q$  be a prime power congruent to 1 mod 4. (Then  $-1$  is a square in the field  $\mathbb{F}_q$ .) The **Paley graph**  $P_q$  has as vertex set the field  $\mathbb{F}_q$ , with an edge from  $x$  to  $y$  if and only if  $y - x$  is a non-zero square in  $\mathbb{F}_q$ . (The remark shows that this is a symmetric relation.)

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The automorphism group of  $P_q$  is the group

$$\{x \mapsto ax^\sigma + b : a, b \in \mathbb{F}_q, a \neq 0, a \text{ square}, \sigma \in \text{Aut}(\mathbb{F}_q)\}.$$

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Hence

- ▶ If  $q$  is prime, then any two points form a base;
- ▶ Otherwise, some well-chosen triples form bases, but if we choose badly we might need as many as  $\sqrt{q} + 1$  points in a base.

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*Let  $G$  be a permutation group with the property that any set of  $k$  points is a minimal base. Then  $G$  is  $(k - 1)$ -transitive. In particular, if  $k \geq 5$ , then  $G$  is sharply  $k$ -transitive (and so is  $S_k$ ,  $S_{k+1}$ ,  $A_{k+2}$ , or  $M_{12}$  for  $k = 5$ ).*

## Digression

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For  $k = 2, 3$  we have Frobenius and Zassenhaus groups respectively.

## Paley graphs

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A different approach was proposed by Evdokimov and Ponomarenko, using the notion of **coherent configuration** (which of course also occurs in Babai's classic proof). This notion was developed by Donald Higman in the west and Boris Weisfeiler in the Soviet Union from the notion of **association scheme** in statistics.

## Coherent configurations

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- ▶ the diagonal is a union of parts of  $C$ ;
- ▶ the converse of a part of  $C$  is a part of  $C$ ;
- ▶ if  $(x, y) \in C_k$ , then the number of  $z \in \Omega$  such that  $(x, z) \in C_i$  and  $(z, y) \in C_j$  depends only on  $i, j, k$  and not on  $x, y$ .

## Coherent configurations, 2

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Hence, given any family  $F$  of subsets of  $\Omega \times \Omega$ , there is a unique finest coherent configuration containing them, which we call the coherent configuration **generated** by  $F$ .

In particular, the partition into singletons forms the “trivial” coherent configuration, which we denote by  $E$ .

## EP-dimension

The **EP-dimension** of a coherent configuration  $C$  is the smallest number  $k$  for which there exist  $k$  points  $a_1, \dots, a_k \in \Omega$  such that the coherent configuration generated by  $C$  and  $(a_1, a_1), \dots, (a_k, a_k)$  is the trivial configuration  $E$ .

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Clearly the EP-dimension of a coherent configuration is not smaller than the base size of its automorphism group, and is not greater than the determining number of the configuration (suitably defined); so it *might* be strong enough for good bounds on base size but simple enough that it can be computed fairly efficiently ...

# Paley graphs

## Conjecture

*Let  $q$  be a prime congruent to 1 (mod 4). Then the EP-dimension of the Paley graph  $P_q$  is 2.*

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Here is how it works for  $p = 13$ . Without loss, choose the potential base  $\{0, 1\}$ . This “distinguishes” four sets of the remaining vertices. The vertices joined to 0 but not 1 are 3, 9, 12, and the induced subgraph is a path  $3 \sim 12 \sim 9$ . So 12 is “distinguished”. Now 12 and 0 distinguish 11, and we can work all the way around.

## The story continues

The EP-dimension of a coherent configuration is sandwiched between the base size of its automorphism group and the determining number. Both inequalities can be strict.

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### Example

Very many, probably “almost all”, strongly regular graphs have trivial automorphism group. Such graphs have base size 0, but the EP-dimension is strictly positive.

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### Example

We certainly know that the EP-dimension of some small Paley graphs of prime degree is 2; but the determining number is about  $\log q$ .