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Sets, Logic and Categories

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Honk Kong Barcelona
Budapest

Preface

Set theory has a dual role in mathematics. In pure mathematics, it is the place where questions about infinity are studied. Although this is a fascinating study of permanent interest, it does not account for the importance of set theory in applied areas. There the importance stems from the fact that set theory provides an incredibly versatile toolbox for building mathematical models of various phenomena.

Jon Barwise and Lawrence Moss, *Vicious Circles: On the Mathematics of Non-Wellfounded Phenomena* [4]

Reasoning and logic are to each other as health is to medicine, or – better – as conduct is to morality. Reasoning refers to a gamut of natural thought processes in the everyday world. Logic is how we ought to think if objective truth is our goal – and the everyday world is very little concerned with objective truth. Logic is the science of the justification of conclusions we have reached by natural reasoning.

Julian Jaynes, *The Origin of Consciousness in the Breakdown of the Bicameral Mind* [24]

Much of Mathematics is dynamic, in that it deals with morphisms of an object into another object of the same kind. Such morphisms (like functions) form categories, and so the approach via categories fits well with the objective of organizing and understanding Mathematics. That, in truth, should be the goal of a proper philosophy of Mathematics.

Saunders MacLane, *Mathematics: Form and Function* [36]

The three subjects in the title of this book all play a dual role in mathematics, which is hinted at in the quotations above. On the one hand, they are usually regarded as part of the foundations on which the structure of mathematics is built. This is obviously very important, philosophically as well as mathematically, since mathematics is commonly regarded as the most securely founded intellectual discipline of all. On the other hand, they are branches of mathematics in their own right: we use standard mathematical techniques to prove theorems in set theory, logic and category theory, and we use results from these areas in other parts of mathematics.

Here is one example of this, chosen from many possible. The Four-Colour Theorem asserts that any map drawn in the Euclidean plane (with reasonable assumptions about the shapes and borders of the countries) can be coloured with four colours in such a way that neighbouring countries are given different colours. This theorem was proved by Appel and Haken [2] with the aid of a very substantial computer calculation. Of course, the computer can only prove the result for *finite maps* (those involving only finitely many countries). But applying the Compactness Theorem of logic, it is a simple matter to deduce the infinite version from the finite.

We now give a brief sketch of what each of the three topics comprises.

Formal logic has two aspects, syntactic and semantic. The syntactic aspect is concerned with explaining which strings of symbols are to be regarded as formulae of a particular system, and which strings are 'theorems'. No meaning is attached to the symbols: the rules for testing whether a string is a well-formed formula, and the rules for manipulating formulae to prove 'theorems', are purely formal, and could be carried out by a computer without any intelligence. The semantic aspect is concerned with attaching meaning to the formulae, so that any formula expresses a mathematical fact (which may be true or false, or sometimes true and sometimes false depending on the values of variables it contains), and any theorem expresses a true fact about the systems to which it applies. We will deal with *first-order logic*, which is close to the actual practice of mathematicians: the structures to which it applies are sets on which various relations and functions are defined (including groups, ordered sets, and so on). We are concerned with the classes of structures satisfying particular formulae, or the theories (sets of formulae) holding in particular structures.

Set theory is the traditional foundation of mathematics. As the last paragraph suggests, the structures which first-order logic describes are based on sets. We are all familiar with definitions like 'A group is a set with a binary operation satisfying

. . . ’. We will indicate, without giving a detailed proof, that anything in mathematics (from natural numbers to probability measures) can be interpreted as a set. On the other hand, set theory is a mathematical subject, and can be interpreted in different ways: different models of set theory support different mathematics. The best-known example of this is the Axiom of Choice, which has important applications in all branches of mathematics.

A more radical re-formulation of the basics is provided by category theory. This begins by emphasizing that it is the transformations between objects, rather than the objects themselves, which are really fundamental in mathematics. If two groups are isomorphic, we do not care that one is a group of matrices and the other a group of permutations. The transformations between structures can be pictured as arrows connecting various dots in a large diagram encompassing the branch of mathematics in question. What we need to know is how these transformations combine. It is possible to lay down rules for this. When this is done, we observe that certain mathematical objects (such as groups) themselves obey these axioms, giving a two-level structure to the subject. It is possible to build on this by considering the category of categories. Moreover, set theory is not really necessary to this foundation; sets and mappings form a category on the same footing as any other.

This book doesn’t begin right at the beginning. It is assumed that you have met basic properties of sets (union, intersection, Venn diagrams, equivalence and order, one-to-one and onto functions), and the number systems (the natural numbers, integers, rational, real and complex numbers). This can be found in an introductory course on pure mathematics or discrete mathematics, such as Geoff Smith’s book [42] in the SUMS series. It will help if you have met some logic before, but this is not vital. In a sense, this book follows on from David Johnson’s book [26] in the series, although there are some differences in notation. Since examples of categories are taken from every branch of mathematics, the more you know the better; but you should at least have met familiar algebraic structures such as groups, rings and vector spaces from an axiomatic approach. This can be found in David Wallace’s book [46] for abstract algebra, or Blyth and Robertson [7] for linear algebra.

Chapter 1 is about sets, from the point of view that we know what a set or collection of elements is. Right from the outset, we see that care is required to avoid such difficulties as Russell’s Paradox. One of the main successes of set theory is to extend the theory of counting to infinite sets. In Chapter 2, we develop the theory of ordinal numbers, which ‘count’ certain kinds of ordered sets (*well-ordered sets*).

The next three chapters are about logic. Chapter 3 introduces the concept of a formal system, and treats propositional logic in detail; we see how the formal deduction and the semantics meet in the Soundness and Completeness Theorem for this logic. In Chapter 4, we do the same job for first-order logic, which treats the objects of mathematics (sets, relations and functions). Chapter 5 takes a few steps into model theory, which is concerned with the relation between the structures of mathematics and the logical formulae they satisfy.

In Chapter 6, we return to set theory, armed with the ideas of first-order logic, and give the Zermelo–Fraenkel axioms. One of these axioms is the (somewhat unintuitive) Axiom of Choice; we examine its consequences for mathematics. We also look at what can be said about classes which are not sets.

Chapter 7 is an account of category theory, the approach which treats functions rather than sets as basic. It is not yet possible to build mathematics on the foundation of category theory rather than set theory, but we look briefly at what can be done.

Finally, we take a very brief look at the philosophy of mathematics, and give some suggestions for further reading.

There is a World Wide Web site associated with this book, at the URL

`http://www.maths.qmw.ac.uk/~pjc/slc/`

This will contain solutions to all the exercises, a list of corrections, and links to related sites of interest on the Web.

The picture on p. 16 is by Neill Cameron.

Contents