Generating a group by a transversal

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Abstract

I answer a question of Vivek Jain by showing that, if H is a core-free subgroup of a finite group G, then there is a transversal for H in G which generates G. The result can be strengthened in a couple of ways: we may assume that the representative of the coset H is the identity; or we may relax the condition that H is core-free (but not too far). The crucial ingredient in the proof is a theorem of Whiston on the maximum size of an independent set in the symmetric group.

Recall that the *core* of a subgroup H of a group G is the largest normal subgroup of G contained in H (the intersection of the conjugates of H in G). Recently, Vivek Jain [1] asked whether the following is true:

Theorem 1 *Let G be a finite group and H be a proper subgroup of G such that the core of H in G is trivial. Then there exist a right transversal of H in G which generates G.*

Proof Let the index of H in G be n. Since H is core-free, G acts faithfully on the cosets of H; so we may assume that G is a subgroup of the symmetric group S_n. We suppose that G is a counterexample to the theorem, and that T is a transversal for G (the point stabiliser) such that $\langle T\rangle$ is as large as possible.

A set S of elements of a group G is *independent* if no element of S is contained in the subgroup generated by all the others. Whiston [2] showed that the maximum size of an independent set in S_n is $n-1$.

Now $|T| = n$, so T cannot be independent; this means that we can choose a set of $n-1$ elements of T which generate the same subgroup. Now replace the missing element by one chosen from the same subgroup of H but not lying in $\langle T\rangle$; the new transversal generates a larger subgroup, contrary to assumption.

In fact, Whiston proved a little more than stated above; he showed that an independent set of size $n-1$ in S_n generates S_{n-1}. This allows us to improve the theorem slightly.

Proposition 2 *Let H be a proper subgroup of the finite group G.*

(a) If H is core-free, then there is a transversal for H in G which generates G, such that the representative of the subgroup G is the identity.

(b) If the core of H in G is cyclic, then there is a transversal for H in G which generates G.

Proof (a) As before, let G be a countererxample, and choose T to be a transversal in which the representative of H is the identity, for which $\langle T\rangle$ is maximal. Then $|T\setminus\{1\}| = n-1$. If this set is not independent, we can generate a larger subgroup as in the preceding proof. Otherwise, Whiston's theorem shows that $G = S_n$ and $H = S_{n-1}$, in which case the required transversal is easily chosen directly.

(b) Let K be the core of H in G; denote images in G/K by overlines. We can choose a transversal T for H in G such that the representative of H lies in K and $\langle\overline{T}\rangle = G/K$. If K is cyclic, replace the representative of H by an element which generates K; the new transversal generates G.

It is clear that the theorem is not true without some extra condition. For example, suppose that H_1 is a core-free subgroup of G_1 of index n, and let X be a group which cannot be generated by n elements. Set $G = G_1 \times X$ and $H = H_1 \times X$; then no transversal for H in G can generate G.

Problem What can be said about the maximum of the numbers m such that, if H is a subgroup of the finite group G of index n such that the core of H in G can be generated by m elements, then there is a transversal for H in G which generates G?

References

[1] Vivek Jain, personal communication.

[2] Julius Whiston, Maximal independent generating sets of the symmetric group, *J. Algebra* **232** (2000), 255–268.