Problems from BCC22

Edited by Peter J. Cameron

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Abstract

These problems were mostly presented at the problem session at the 22nd British Combinatorial Conference at St Andrews, on 10 July 2009. I have removed two problems that were solved before or during the session; problems which have been subsequently solved are retained and given a number which should not change and can be used to refer to them. Solved problems will not appear in the published version but will remain in this document with some indication of the solution.

For uniformity I have used the British spelling of words like 'colouring' — I hope there are no strong objections to this.

What I would like proposers to do is

- supply me with relevant references;
- check that I have formulated your problem correctly (or reformulated it, in two cases where the original problem has been solved)
- any other relevant comments.

Thanks for your help, and for your contributions!

Problems on block designs

For the problems below, a *block design* can be regarded as a set of v *points* or *treatments*, with a collection of b subsets called *blocks*, each of cardinality k. Its *incidence matrix* is the $v \times b$ matrix (where b is the number of blocks) having (i, j) entry 1 if the ith point is contained in the jth block, and 0 otherwise.

A 2-*design* is a block design with the property that any two points lie in exactly λ blocks, for some $\lambda > 0$. A 2-design with $k = 3$ and $\lambda = 1$ is called a *Steiner triple system*.

Discussion of various optimality criteria for block designs appears in the paper by R. A. Bailey and P. J. Cameron in the conference invited talks volume, cited below. They depend on the *Laplacian eigenvalues* of the design, which are the non-zero eigenvalues of the $v \times v$ matrix whose (i, j) entry for $i \neq j$ is the number of blocks containing points i and j, and whose diagonal entries are chosen so that the row sums are zero (so there is a *trivial eigenvalue* 0, associated with the all-1 eigenvector (see the paper of W. Haemers in the conference volume). Specifically, A- and D-optimality involve respectively maximizing the harmonic mean and the geometric mean of the non-trivial eigenvalues.

Problem BCC 22.1 Nowhere-zero 5-flows for 2-designs Presented by S. Akbari, G. B. Khosrovshahi, A. Mofidy
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A *flow* for a 2-design with v points and b blocks is a vector in the null space of the $v \times b$ incidence matrix A of the design, that is, a vector $v \in \mathbb{R}^b$ satisfying $Av = 0$. It is *nowhere-zero* if none of the entries are zero.

It is known that a 2-design with $b > v$ possesses a nowhere-zero flow. [REF??]

Conjecture: A 2-design with $b > v$ possesses a flow with entries from the set $\{\pm 1, \pm 2, \pm 3, \pm 4\}$.

References

[1] S. Akbari, K. Hassani Monfared, M. Jamaali, E. Khanmohammadi and D. Kiani On the existence of nowhere-zero vectors for linear transformations *Bull. Austral. Math. Soc.*, to appear.

Problem BCC 22.2 Infinite perfect Steiner triple systems Presented by M. J. Grannell, T. S. Griggs and B. S. Webb
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Let S be Steiner triple system on set V of v points. For any $a, b \in b$, define a graph $G_{a,b}$ with vertex set $V \setminus \{a, b, c\}$, where $\{a, b, c\}$ is the unique block containing a and b, as follows: x is adjacent to y if and only if either $\{a, x, y\}$ or $\{b, x, y\}$ is a block. The graph $G_{a,b}$ has valency 2 on $v-3$ vertices, and all cycles have even length. We say that S is *uniform* if all graphs $G_{a,b}$ are isomorphic, and *perfect* if each consists of a single cycle. Only finitely many perfect finite Steiner triple systems are known.

Problem: Does there exist an infinite perfect Steiner triple system? Such a system can be no larger than countably infinite, since an infinite "cycle" is a two-way infinite path and has only countably many vertices.

Infinite uniform systems are constructed in [1], where each graph $G_{a,b}$ consists of infinitely may two-way infinite paths.

References

[1] K. M. Chicot, M. J. Grannell, T. S. Griggs and B. S. Webb, On sparse countably infinite Steiner triple systems, *J. Combinatorial Designs* **18** (2010), 115–122.

Problem BCC 22.3 A- and D-optimality for block designs Presented by R. A. Bailey

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The following theorem is discussed in [1], where more information about optimality is given. Briefly, a block design is D-optimal (among designs in a given class) if its concurrence graph maximises the number of spanning trees, and is A-optimal if its concurrence graph maximises the sum of resistances between pairs of terminals (when each edge is a 1-ohm resistor).

Theorem Let $k = 2$, and fix $c \geq 0$. Then there is a threshold v_0 such that, among block designs with $b = v + c$, if $v > v_0$, then the D-optimal designs are very different from the A-optimal designs.

Problem

- Can we replace $b = v + c$ by $b = cv$?
- Can we replace $k = 2$ by general k?
- Can we replace the A-criterion by

$$\left(\frac{1}{v-1}\sum_{i=1}^{v-1}\theta_i^{-p}\right)^{1/p}$$

  for $p > 0$, where $\theta_1, \ldots, \theta_{v-1}$ are the non-trivial Laplacian eigenvalues of the design?
- If so, is v_0 a monotonic function of p?
- Be more precise about 'very different'.

References

[1] R. A. Bailey and P. J. Cameron, Combinatorics of optimal designs, in *Surveys in Combinatorics 2009* (ed. S. Huczynska, J. D. Mitchell and C. M. Roney-Dougal), London Math. Soc. Lecture Notes **365**, Cambridge University Press, Cambridge, 2009, pp. 19–73.

[2] W. Haemers, Regularity and the spectra of graphs, in *Surveys in Combinatorics 2009* (ed. S. Huczynska, J. D. Mitchell and C. M. Roney-Dougal), London Math. Soc. Lecture Notes **365**, Cambridge University Press, Cambridge, 2009, pp. 75–90.

Problems on Latin squares

A *Latin square* of order n is an $n \times n$ array containing n distinct symbols, such that each symbol occurs precisely once in each row and once in each column. Usually we assume that the symbol set is $\{1, \ldots, n\}$, with the same set labelling rows and columns.

A *subsquare* of a Latin square is a set of k rows and k columns such that the corresponding k^2 entries contain just k distinct symbols (forming a Latin square in its own right). It is known that the order of a proper subsquare is at most half of the order of the equare.

An *autotopism* of a Latin square L is a triple (α, β, γ) of permutations of $\{1, \ldots, n\}$ such that, if the (i, j) entry of L is k, then the (i^α, j^β) entry of L is k^γ. The autotopisms of a Latin square form a group, the *autotopism group* of the square.

A Latin square of order n is "equivalent" to an edge-colouring of the complete bipartite graph $K_{n,n}$ with n colours: if L has (i, j) entry k, we give colour k to the edge joining the ith vertex in the left-hand bipartite block to the jth vertex in the right-hand block.

Problem BCC 22.4 Squared Latin squares Presented by Graham Farr
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Does there exist a *squared Latin square*, that is, a Latin square partitioned into Latin subsquares, all of distinct sizes?

Note that the cells of the Latin subsquares do not have to be contiguous in the main Latin square. So we are not talking about a geometric tiling of the main square. Indeed, this freedom is essential, if we are to have any hope: a short argument due to Ian Wanless shows that no squared Latin square can be based on a "squared square" (i.e., a tiling of a square by smaller squares, all of distinct sizes — see Brooks, Smith, Stone & Tutte [1]).

Douglas Stones has observed that any squared Latin square must have order at least 21, based on the fact that any proper Latin subsquare of a Latin square of order n must have order at most $n/2$. Probably, if examples exist, they are significantly larger than this.

References

[1] R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, The dissection of rectangles into squares, *Duke Math. J.* **7** (1940), 312–340.

Problem BCC 22.5 Which square has most autotopisms? Presented by Douglas Stones
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Which Latin squares of given order n have the largest autotopism group? For which n is this maximum achieved by a group table?

Editor's note: R. A. Bailey has provided me with the following comments on this problem, which show that a group table does not always achieve the maximum.

We represent a Latin square of order n by a collection of triples (a, b, c), where a, b and c are elements of a set S of size n. Any two entries in the triple determine the third.

The Cayley table of a group G is a Latin square over the set G whose triples are (a, b, ab) for a, b in G. It was shown in [4] that the autotopism group of such a Latin square is $(G \times G) \rtimes \mathrm{Aut}(G)$, being generated by permutations of the form $(a, b, ab) \mapsto (ga, bh, gabh)$ for g, h in G and $(a, b, ab) \mapsto (a\alpha, b\alpha, (ab)\alpha)$ for α in $\mathrm{Aut}(G)$. Thus one source of Latin squares with large autotopism groups is the collection of Cayley tables of groups with large automorphism groups. Relative to their size, the groups with the largest automorphism groups are the elementary Abelian p-groups for primes p: if $|G| = p^m$ then $|\mathrm{Aut}(G)| = (p^m - 1)(p^m - p) \cdots (p^m - p^{m-1})$.

Large size of a group of permutations of a fixed set is sometimes associated with high transitivity. The autotopism group of a Latin square must have at least four orbits on unordered pairs of cells if $n \geq 3$. It was shown in [1] that this minimum is achieved if and only if the square is the Cayley table of either an elementary Abelian 2-group of order at least 4 or the cyclic group of order 3.

A source of Latin squares whose autotopism groups are not transitive on S^2 but may be large nonetheless is the collection of Steiner triple systems. If S is the point-set of such a system, then the triples are (a, a, a) for a in

S, and (a, b, c) whenever $\{a, b, c\}$ is a triple. The autotopism group of the corresponding square contains the group of automorphisms of the Steiner triple system, acting as quasigroup automorphisms (but may be larger).

If $n = 2^m - 1$ then there is a Steiner triple system formed by the lines in projective $(m-1)$-dimensional space over GF(2). The automorphism group of the Steiner system has order $(2^m - 1)(2^m - 2) \cdots (2^m - 2^{m-1})$. For example, take $n = 15$. There is a Latin square of order 15 defined by the projective space of dimension 3 over GF(2). Its autotopism group has order at least $15 \times 14 \times 12 \times 8$. The only group of order 15 is C_{15}, whose automorphism group has order 8, so the autotopism group of the Cayley table of C_{15} has order $15 \times 15 \times 8$, which is smaller than $15 \times 14 \times 12 \times 8$. So when $n = 15$ the Latin square with the largest autotopism group is not a Cayley table. This may well generalize to infinitely many numbers of the form $2^m - 1$.

A Latin square made from a Steiner triple system of order n is idempotent, in that the diagonal triples are all of the form (a, a, a). Such a square can be expanded to a unipotent Latin square of order $n + 1$ by adjoining ∞ to S, removing all triples of the form (a, a, a), inserting triples (∞, a, a), (a, ∞, a) and (a, a, ∞) for a in S, and inserting the triple (∞, ∞, ∞). The autotopism group of the new square is at least as large as that of the original.

Taking S to be the Steiner system whose triples are the lines in affine 4-dimensional space over GF(3), we obtain a Latin square of order 82 whose autotopism group has order at least $81 \times 80 \times 78 \times 72$. The only groups of order 82 are C_{82} and D_{82}, whose automorphism groups have order 40 and 41×40 respectively. As $81 \times 80 \times 78 \times 72 > 82 \times 82 \times 41 \times 40$, the Latin square of order 82 with the largest autotopism group is not a Cayley table. Again, this may hold for infinitely numbers of the form $9^r + 1$.

If L and M are both Latin squares, then $\mathrm{Aut}(L \otimes M)$ contains $\mathrm{Aut}(L) \times \mathrm{Aut}(M)$. This observation gives us a third class of examples. For example, taking L to be the square formed from the Steiner triple system defined by projective 4-dimensional space over GF(2) and M to have order 2 gives a Latin square of order 62 whose autotopism group has order at least $(31 \times 30 \times 28 \times 24 \times 16) \times (2 \times 2)$. The only groups of order 62 are C_{62} and D_{62}, whose automorphism groups have order 30 and 31×30 respectively. Thus neither Cayley table has autotopism group as large as that of $L \otimes M$.

Suppose that L is a Latin square of order n whose autotopism group is larger than the autotopism group of any Cayley table of a group of order n. Suppose further that n is a power of a prime q, and that p is a prime which is not congruent to 1 modulo q and which does not divide the order of the automorphism group of any group of order n. Then any group of order np has the form $G \times C_p$, with automorphism group $\mathrm{Aut}(G) \times C_{p-1}$. If M is the Cayley table of C_p, then $\mathrm{Aut}(L \otimes M) \geq \mathrm{Aut}(L) \times ((C_p \times C_p) \rtimes C_{p-1})$, which is strictly greater than the size of the autotopism group of the Cayley table for any group of order np.

Thus there are three methods—Steiner systems, extending an idempotent Latin square to a unipotent Latin square, and taking direct products—that give large numbers of values of n for which the Latin square with the largest autotopism group is not a Cayley table.

References

[1] R. A. Bailey, Latin squares with highly transitive automorphism groups, *Journal of the Australian Mathematical Society, Series A* **33** (1982), 18–22.

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[3] J. Dénes and A. D. Keedwell, *Latin Squares and their Applications*, English Universities Press Limited, 1974.

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Problem BCC 22.6 Edge-colourings of complete bipartite graphs with no short cycles Presented by P. Dukes
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Let n be an integer greater than 1. Define $l(n)$ to be the minimum, over all proper n-edge colourings of $K_{n,n}$, of the longest 2-coloured cycle in such a colouring.

Problem Is $l(n) \leq 6$ for all sufficiently large n?

Cameron [1] showed that $l(n) = 4$ if and only if $n = 2^k$. Ninčák and Owens [2] showed that $l(n) \leq 2n - 4$ (with a few small exceptions); the upper bound was reduced to a polynomial in $\log n$ by Dukes and Ling [3], and subsequently to 1720 by the same authors [4].

References

[1] P. J. Cameron, Minimal edge-colourings of complete graphs. *J. London Math. Soc.* **11** (1975), 337–346.

[2] J. Ninčák and P. Owens, On a problem of R. Häggkvist concerning edge-colouring of bipartite graphs, *Combinatorica* **24** (2004), 325–329.

[3] P. Dukes and A. C. H. Ling, Edge-colourings of $K_{n,n}$ with no long two-coloured cycles, *Combinatorica* **28** (2008), 373–378.

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Finite geometry

Problem BCC 22.7 Hemisystems in Hermitian varieties Presented by Frédéric Vanhove
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A *Hermitian variety* $H(2n+1, q^2)$ is the set of isotropic points in the projective space $\mathrm{PG}(2n+1, q^2)$ of a Hermitian form (a non-degenerate alternating bilinear form) on the underlying vector space $\mathrm{GF}(q)^{2n+2}$. A *generator* is a projective subspace of maximum possible dimension n contained in the Hermitian variety.

Problem 1 When does a Hermitian variety $H(2n+1, q^2)$ possess a set G of generators such that, for some constant λ, every $(n-1)$-dimensional space is incident with exactly λ elements of G?

The residue of any $(n-2)$-dimensional space in $H(2n+1, q^2)$ is isomorphic to $H(3, q^2)$. In this residue, such a set G would induce a set of lines, covering every point λ times. A famous result by Segre [2] implies that q must be odd and that $\lambda = (q+1)/2$.

Moreover, the residue of any $(n-3)$-dimensional space in $H(2n+1, q^2)$ is isomorphic to $H(5, q^2)$, where such a set G would induce a set of planes, covering every line $(q+1)/2$ times. Hence one might start by considering the following case of the original problem:

Problem 1a When does a Hermitian variety $H(5, q^2)$ possess a set G of planes, such that every line is incident with exactly $(q+1)/2$ planes of G?

As far as I know, no such sets of planes exists in $H(5, q^2)$, although there are sets of planes for q odd, covering every point the same number of times and of the right size, but not covering lines the same number of times.

Problem 2 Prove in a geometric way that, if S is a partial spread in $H(4n+1, q^2)$ (a set of pairwise disjoint generators) with $|S| = q^{2n+1}+1$, and if G is a set of generators covering every $(2n-1)$-dimensional space exactly $(q+1)/2$-times, that $|S \cap G| = |S|/2 = (q^{2n+1}+1)/2$.

References

[1] J. W. P. Hirschfeld and J. A. Thas, *General Galois Geometries*, Oxford University Press, New York, 1991.

[2] B. Segre, Forme e geometrie Hermitiane con particolare riguardo al caso finito, *Ann. Mat. Pura Appl.* **70** (1965), 1–201.

Graph theory: matchings and colourings

Graph theory definitions should be standard. The *Paley graph* $P(q)$, where q is a prime power congruent to 1 (mod 4), has as vertex set the finite field $\mathrm{GF}(q)$, with vertices x and y adjacent if and only if $y-x$ is a non-zero square in the field.

Problem BCC 22.8 Spectral characterization of perfect matching?
Presented by Willem Haemers
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Do there exist two regular graphs G_1 and G_2 with the same spectrum, such that G_1 possesses a perfect matching (a set of edges covering each vertex once) but G_2 does not?

A positive answer would show that the existence of a perfect matching is not determined by the spectrum of any of several variants of the adjacency matrix of the graph (such as the Laplacian or Seidel matrices).

A pair of non-regular graphs with the above property has been found by Aidan Roy. The second is obtained from the first by Godsil–McKay switching [1].

See Haemers' paper in the invited speakers' volume, cited above under Problem BCC22.3.

References

[1] C. D. Godsil and B. D. McKay, Constructing cospectral graphs, *Aequationes Math.* **25** (1982), 257–268.

Problem BCC 22.9 Colouring vertex-transitive graphs Presented by Rong Gao and David Penman

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Let G be a (finite, undirected, simple) graph with n vertices and independence number $\alpha(G)$, chromatic number $\chi(G)$ and choosability (list-chromatic number) $\mathrm{ch}(G)$. It is well-known that $\mathrm{ch}(G) \geq \chi(G) \geq n/\alpha(G)$ for any graph. This problem focuses on how close to each other these three quantities have to be. We know, by a famous result of Bollobás [3], and an extension due to Kahn (see Alon [1, Proposition 4.4]) that for random graphs $G(n, 1/2)$ we **whp** have that these three quantities are all $(n/(2\log_2(n)))(1 + o(1))$.

Specifically, we consider vertex-transitive graphs G from now on. Here it is (quite) well-known that we have an upper bound (say)

$$\chi(G) \leq 1 + n\log(n)/\alpha(G). \tag{1}$$

(Here logs are to the base e). See Babai's article [2] in the Handbook of Combinatorics, or the 'Symmetric Hypergraph Theorem', in Graham, Rothschild and Spencer's book [5] on Ramsey Theory.

Question 1 Does the same upper bound hold for the choosability of vertex-transitive graphs?

Note that the proof of (1) in Babai's paper considers random translates of a maximum independent set, assumed to be coloured with one colour, and for choosability we cannot assume the existence of such a set. Complete bipartite graphs $K_{r,r}$ have (for large n) $n\log(n)/\alpha \simeq 2\log(n)$ and $\mathrm{ch}(G) \simeq \log_2(n)$ so if (1) is still true for choosability it cannot be improved by more than a (small) multiplicative constant. There is a curious tension between vertex-transitive ('looking the same at all vertices') and choosability ('(very probably) not looking the same at all vertices') which may make this difficult. Even if the bound itself is not true, close approximations to it might well still be of interest.

An earlier student of the second proposer (Eleni Maistrelli) observed that Kneser graphs with vertex set the k-element subsets of $\{1, 2, \ldots m\}$ and two vertices adjacent if and only if the corresponding sets are disjoint, has (for $m > 2k$, say) $\binom{m}{k}$ vertices, independence number $\binom{m-1}{k-1}$ by the Erdős–Ko–Rado theorem [4] but $\chi = m - 2k + 2$ by a result of Lovász [6] (using topology). As Kneser graphs are vertex-transitive, taking $k = m/\omega(m)$ for ω tending to infinity (slowly) with m, we see on tidying up that (1) cannot be much improved in general. On the other hand, there are vertex-transitive graphs for which $\chi = n/\alpha$, e.g. Paley graphs of square order.

Question 2 (deliberately open-ended). Investigate the bound (1) further: for example, prove theorems of the form

> If G is a vertex-transitive graph with certain additional properties, then $\chi(G) \leq n \log(n)/(\alpha(G) f(n))$

(the bigger $f(n)$ can be, the better in some sense), and/or giving further classes of vertex-transitive graphs for which the chromatic number is close to the upper bound (1).

An obvious starting point might be to obtain or estimate the choosability of suitable Kneser graphs (which does not look easy to the proposers).

Some motivation for the questions comes from the first proposer's Ph.D thesis (supervised by the second proposer), where (1), together with results from J. Williford's Ph.D thesis [8, 7] was used to show that the Erdős–Rényi polarity graphs ER_q, for q a power of 2, have chromatic number at most $4\sqrt{2}\log(q)\sqrt{q}(1+o(1))$ as $q \to \infty$. Since the lower bound n/α has order of magnitude $\sqrt{q}$ for (any) ER_q, it would be interesting to see if one can get the upper bound down to $K\sqrt{q}$ for some constant K. Answers to the first part of Question 2 might help with this.

References

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[2] L. Babai, Automorphism groups, isomorphism, reconstruction, *Handbook of combinatorics*, 1447–1540, Elsevier, Amsterdam, 1995.

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[4] P. Erdős, Chao Ko, and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* (2) **12** (1961), 313–320.

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[8] J. Williford, PhD thesis, University of Delaware, 2004.

Problem BCC 22.10 Colourful paths Presented by S. Akbari, F. Khaganpoor, S. Moazzeni
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Let G be a graph, and c a proper vertex colouring of G with $\chi(G)$ colours. A *colourful path* is a path with $\chi(G)$ vertices such that every colour appears on the vertices of the path exactly once.

It is known [1] that, if G is a connected graph not equal to C_7, and v a vertex of G, then there is a proper vertex colouring of G with $\chi(G)$ colours containing a colourful path starting at v.

Problem If G is a connected graph not equal to C_7, then there is a proper vertex colouring of G with $\chi(G)$ colours such that, for every vertex v, there exists a colourful path starting at v.

References

[1] S. Akbari, V. Liaghat and A. Nikzad, Colorful paths in vertex-colorings of graphs, preprint, Sharif University, 2009.

Problem BCC 22.11 Dynamic list colouring Presented by M. Ghanbari, S. Akbari and S. Jahanbekam
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A proper vertex colouring of a graph G is called a *dynamic colouring* if for every vertex v of degree at least 2, the neighbours of v receive at least two different colours. The smallest number of colours required for a dynamic colouring of G is called the *dynamic chromatic number* of G, denoted $\chi_2(G)$.

Let G be a graph, and, for every vertex v, let $L(v)$ be a list of colours. An *L-colouring*, *list colouring* or *choice function* is a proper colouring f of G with $f(v) \in L(v)$ for all vertices v. The graph is *k-choosable* if every assignment of k-element lists to the vertices permits a list clouring. The *list chromatic number*, *choice number*, or *choosability* of G, denoted $\mathrm{ch}(G)$, is the minimum number k such that G is k-choosable.

A list colouring which is dynamic is called a *dynamic L-coouring* The graph G is called *k-list dynamic colourable* if every assignment of k-element lists to the vertices permits a dynamic colouring. The *list dynamic chromatic number* of G, denoted $\mathrm{ch}_2(G)$, is the smallest number k for which the graph is k-list dynamic colourable.

The proposers conjectured that, for any graph G,

$$\mathrm{ch}_2(G) = \max(\mathrm{ch}(G), \chi_2(G).$$

A counterexample was proposed by O. Riordan. We build it as follows. Take a triangle $\{a, b, c\}$. Take two copies of $K_{3,3}$; join all vertices of the first copy to b, and all vertices of the second to c. Call this graph H. Now take two copies of H, and a new vertex z which is joined to the two vertices labelled a in the two copies of H; call the resulting graph G. This graph has $\chi_2(G) = \mathrm{ch}(G) = 3$ but $\mathrm{ch}_2(G) > 3$. A document containing the proof is available from the editor.

An alternative question might be:

Problem Is it true that $\mathrm{ch}_2(G)$ is bounded above by a function of $\chi_2(G)$ and $\mathrm{ch}(G)$, for any graph G?

Other graph-theoretic problems

Problem BCC 22.12 Spanning trees of cubic graphs Presented by Arthur Hoffmann-Ostenhof
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Let us call a cubic graph which has a spanning tree such that all of its vertices are of degree 1 or 3, a *green graph*. For example, K_4, the Petersen graph, and every Halin graph is a green graph, whereas the cube is not a green graph. We do not aim to characterize green graphs. In contrast, we claim that all connected cubic graphs have a nice property in common with all green graphs, which is stated in the conjecture below. (A vertex of degree 1 in a tree is called a leaf.)

Conjecture Every connected cubic graph G has a spanning tree such that its set of leaves induces a 2-regular subgraph of G.

One might prove this conjecture by solving the following problem which is interesting for its own sake:

Problem Find a natural and sufficient condition such that a connected graph H which has only vertices of degree 2 and 3, has a matching M such that $H - M$ is a tree.

Problem BCC 22.13 Synchronizing graphs Presented by Eduardo Canale
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Given a graph $G = (V, E)$, a *unit representation* of G is a function ρ from the vertex set of G to the unit sphere in $\mathbb{R}^n$ (so $\|\rho(v)\| = 1$ for all $v \in V$). The *energy* of ρ is

$$\mathcal{E}(\rho) = \sum_{vw \in E} \|\rho(v) - \rho(w)\|^2.$$

We say that G is *synchronizing* if, for $n = 2$, every local minimum of $\mathcal{E}(G)$ is a global minimum.

Problem Classify the synchronizing graphs.

Remark A solution would have application to coupled oscillators in dynamical systems: see [1].

References

[1] E. Canale and P. Monzón, Almost global synchronization of symmetric Kuramoto coupled oscillators, *Systems Structure and Control*, InTech Education and Publishing (2008), pp. 167–190.

Problem BCC 22.14 Bases for Paley graphs Presented by Robert Bailey and Peter Cameron

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Consider the following 'partition refinement' procedure for graphs. We begin with a graph G on the vertex set V, and a partition π_0 of V. In one step, we are allowed to do the following:

- choose a subset Y of V which is a union of parts of the current partition π;
- refine the current partition according to the degrees of vertices in the induced subgraph of G on Y.

Problem Let G be the Paley graph $P(p)$ on p vertices, where p is a prime congruent to 1 (mod 4). Let π_0 be the partition with two singleton parts and all other vertices in a single part. Is it possible to reach the partition into singletons by the above partition refinement?

Example Let $G = P(13)$, and $\pi_0 = \{\{0\}, \{1\}, \{2, \ldots, 12\}\}$. By taking $Y = \mathbb{Z}_{13} \setminus \{0\}$ and then $Y = \mathbb{Z}_{13} \setminus \{1\}$, we can reach a partition π having the set of vertices joined to 1 but not 0 as a part. The induced subgraph is a path on $\{11, 2, 5\}$, so we obtain a partition with $\{2\}$ as a part. Then working round the cycle we 'fix' every part.

Remark This procedure will not split orbits of the subgroup of the automorphism group fixing the chosen vertices. So the answer would be 'no' for a Paley graph with order a proper power of a prime.

Editor's Remark This problem has been solved: see [1].

References

[1] S. Evdokimov, M. Karpinski and I. Ponomarenko, On a new high-dimensional Weisfeiler–Lehman algorithm, *J. Algebraic Combinatorics* **10** (1999), 29–45.

Problem BCC 22.15 Circuits in cubic bridgeless graphs Presented by Roland Häggkvist
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These three problems share a theme but are succesively more difficult. Let G be a cubic bridgeless graph.

Problem 1 Prove that G contains a pair of circuits C and Q such that

- $E(C) \cap E(Q)$ is a matching;
- $G \setminus (E(C) \cap E(Q))$ is bridgeless.

Problem 2 Prove that for every circuit C, there exists a circuit Q such that, for a fixed edge e in C,

- $E(C) \cap E(Q)$ is a matching;
- $G \setminus (E(C) \cap E(Q))$ is bridgeless;
- $e \in E(C) \cap E(Q)$.

Problem 3 Prove that, for every circuit C, and every pair e_1, e_2 of consecutive edges of C, there exist a pair of circuits Q_1, Q_2 such that

- $M_1 = E(C) \cap E(Q_1)$, $M_2 = E(C) \cap E(Q_2)$ and $M_3 = E(Q_1) \cap E(Q_2)$ are matchings;
- $e_1 \in E(Q_1$ and $e_2 \in E(Q_2)$;
- $G \setminus (M_1 \cup M_2 \cup M_3)$ is bridgeless.

Problem BCC 22.16 Domination polynomial Presented by G. Aalipour, S. Akbari, Z. Ebrahimi
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A *dominating set* in a simple graph G is a set S of vertices of G such that every vertex of G is either in S or adjacent to a vertex in S. The *domination polynomial* of G iis the polynomial

$$D(G, x) = \sum_{i=0}^{n} d(G, i) x^i,$$

where $d(G, i)$ is the number of dominating sets of cardinality i in G, and n the number of vertices of G. Call two graphs G and H *D-equivalent* if $D(G, x) = D(H, x)$.

It is known that the D-equivalence class of the complete bipartite graph $K_{n,n}$ contains only $K_{n,n}$ and $K_n \times K_2$. Similarly the D-equivalence class of $K_{n,n-1}$ contains just two graphs up to isomorphism.

Problem Show that the D-equivalence class of $K_{m,n}$ consists only of $K_{m,n}$ if $n - m \geq 2$.

Problem BCC 22.17 Eating trees by cutting leaves Presented by Piotr Micek and Bartosz Walczak
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Alice and Bob share a tree, whose vertices have non-negative weights. Starting with Alice, they alternately cut leaves (one in each move) and collect their weights. Both want to maximize their final gain.

There are trees on which the guaranteed outcome for Alice is zero, for example the path of length 2 with weights zero on both leaves and 1 on the intermediate vertex. But surprisingly, if the tree has an even number of vertices, Alice can guarantee herself at least 1/4 of the total weight [7].

Conjecture Alice can guarantee herself at least 1/2 of the total weight of any tree with an even number of vertices.

Note that the fraction 1/2 is best possible. The conjecture is true for trees which are subdivisions of stars.

References

[1] P. Micek and B. Walczak, Parity in graph sharing games, preprint, Jagiellonian University, 2010.

Problem BCC 22.18 Colouring intervals Presented by Piotr Micek
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Consider the following game between two players Spoiler and Algorithm.

- Spoiler presents, one-by-one, intervals on the real line, in such a way that the maximum number of pairwise intersecting intervals (that is, the clique number of the corresponding interval graph) never exceeds w;
- Algorithm colours incoming intervals in such a way that intersecting intervals receive different colours.

Problem How many colours does Algorithm require to perform the task?
It is known [1] that the number required is between $3w/2$ and $2w - 1$.
A variant allows proper intervals of arbitrary length.

References

[1] B. Bosek, K. Kloch, T. Krawczyk and P. Micek, On-line version of Rabinovitch theorem for proper intervals, preprint, Jagiellonian University, 2009.

Problem BCC 22.19 Removing sets of edges Presented by Carrie Rutherford, Robin Whitty
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Which subsets of edges can be removed from the complete graph K_n and still allow every unlabelled n-vertex tree to be embedded in what remains? In particular, do any two maximal such sets have the same cardinality?

The maximum size of such edge sets was settled asymptotically by Chung and Graham in 1979 [1] (the proof is given in [2]) but the presenters find no mention of the structural issue.

References

[1] Chung, F.R.K. and Graham, R.L., "On universal graphs", *Annals of the New York Academy of Sciences*, 319, 1979, 136–140.

[2] Chung, F.R.K. and Graham, R.L, "On universal graphs for spanning trees", *J. London Math. Soc.* (2), 27, 1983, 203–211.

Problem BCC 22.20 A problem on subsets of cycles Presented by Nicolas Lichiardopol
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Positive integers $a, b, \alpha, \beta, \gamma$ are given. What is the smallest size $L(a, b, \alpha, \beta, \gamma)$ of a cycle C whose vertex set has subsets A and B of cardinalities a and b respectively such that

- $d_C(x, y) \geq \alpha$ for distinct vertices $x, y \in A$;
- $d_C(x, y) \geq \beta$ for distinct vertices $x, y \in B$;
- $d_C(x, y) \geq \gamma$ for distinct vertices $x \in A$, $y \in B$.

Here d_C denotes the distance in the cycle C.

The presenter (unpublished) has shown that

$$L(a, a, 2, 2, \gamma) = \min(a\gamma, 4a + 2\gamma - 4).$$

Problems on groups and rings

Problem BCC 22.21 Abelian groups defined by triangulations Presented by Ian Wanless
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Let T be a face 2-coloured triangluation of the plane with finitely many triangles, coloured black and white so that triangles sharing an edge have different colours. On each vertex, place an indeterminate x_i, and assume that these commute. Define G_W be the abelian group generated by the elements x_i, subject to the relations that the sum of the indeterminates on the vertices of each white triangle is zero. Let G_B be defined similarly, using the black triangles.

It is known (see Cavenagh and Wanless [1]) that the groups G_W and G_B have the same free rank, and their torsion subgroups have the same order.

Problem Show that G_W and G_B are isomorphic.

References

[1] N. J. Cavenagh and I. M. Wanless, Latin trades in groups defined on planar triangulations, *J. Algebraic Combinatorics*, to appear.

Problem BCC 22.22 The power graph of a group Presented by Shamik Ghosh

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Let G be a group. The *power graph* of G is the graph $\mathcal{G}(G)$ with vertex set G, in which vertices x and y are adjacent if and only if either $x = y^m$ or $y = x^m$ for some natural number m.

This definition suggests two natural questions:

- Clearly $G_1 \cong G_2$ implies $\mathcal{G}(G_1) \cong \mathcal{G}(G_2)$. When does the converse hold?

- Clearly the automorphism group of G is contained in the automorphism group of $\mathcal{G}(G)$. When does equality hold?

The answer to the first question is negative for infinite abelian groups, as the presenter pointed out. Let C_{p^∞} be the group of rational numbers with p-power denominators mod 1, where p is prime. Then $\mathcal{G}(C_{p^\infty})$ is a countably infinite complete graph, independent of the chosen prime.

It is also false for finite groups. Let G be a finite group of exponent 3, that is, satisfying $x^3 = 1$ for all $x \in G$. Then $\mathcal{G}(G)$ consists of $(|G| - 1)/2$ triangles sharing a common vertex (the identity). The elementary abelian group (the direct product of cyclic groups of order 3) has exponent 3, but there are non-abelian groups as well: the smallest is the group of order 27 with presentation

$$G = \langle x, y \mid x^3 = y^3 = [x, y]^3 = 1\rangle,$$

where $[x, y]$ is the commutator $x^{-1}y^{-1}xy$.

In fact, the phenomenon is not uncommon. A short calculation with GAP [3] and its package GRAPE [4] revealed that there are two pairs of groups of order 16 with isomorphic power graphs, and two pairs of groups of order 27. The phenomenon becomes more common; for order 32 there is one quadruple, two triples, and eight pairs.

The presenter asked whether the power graph of an abelian group characterises the group. We have shown that, if G_1 and G_2 are abelian and $\mathcal{G}(G_1) \cong \mathcal{G}(G_2)$, then $G_1 \cong G_2$. We have replaced the original question by the following:

Conjecture Let G_1 and G_2 be finite groups whose power graphs are isomorphic. Then G_1 and G_2 have equally many elements of each order.

This has been checked for groups of order less than 128.

On the second question, we have proved that the only finite group G for which $\mathrm{Aut}(G) = \mathrm{Aut}(\mathcal{G}(G))$ is the Klein group of order 4.

A preprint containing these results is available [2].

Editor's remark The conjecture has been proved, see [1], where it is shown that if the undirected power graphs of two finite groups are isomorphic, then the directed power graphs (with an arc from a to b if b is a power of a) are also isomorphic. It is easy to see that the directed power graph of a group determines the numbers of elements of each order.

References

[1] P. J. Cameron, The power graph of a finite group, II, *J. Group Theory*, in press.

[2] P. J. Cameron and S. Ghosh, The power graph of a finite group, *Discrete Math.*, in press.

[3] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008. `http://www.gap-system.org`

[4] L. H. Soicher, The GRAPE package for GAP, Version 4.3, 2006. `http://www.maths.qmul.ac.uk/ leonard/grape/`

Problem BCC 22.23 A non-invertibility graph Presented by S. Akbari, M. Jamaali, S. A. Seyed Fakhari

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Let F be a field with characteristic not 2. Consider a graph with vertex set $GL_n(F)$ (the group of invertible $n \times n$ matrices over F), and join two matrices A and B if $A + B$ is singular.

It is proved [1] that the clique number of this graph is finite.

Problem Is the vertex chromatic number of this graph finite?

References

[1] S. Akbari, M. Jamaali and S. A. Seyed Fakhari, The clique numbers of regular graphs of matrix algebras are finite, *Linear Algebra Appl.* **431** (2009), 1715–1718

Number-theoretic problems

Problem BCC 22.24 Generating units by arithmetic progressions
Presented by Peter Cameron and Donald Preece
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Let p be a prime and $k \geq 3$. A *k-AP decomposition* of the group $\mathbb{U}_p$ of units of $\mathbb{Z}_p$ (the integers modulo p) is a sequence $(x_1, \ldots, x_k)$ of non-identity elements of $\mathbb{U}_p$ such that

- $(x_1, \ldots, x_k)$ is a k-term arithmetic progression in $\mathbb{Z}_p$ (that is, the differences $x_{i+1} - x_i$ are constant for $i = 1, \ldots, k-1$);
- $\mathbb{U}_p$ is the direct product of the cyclic groups generated by $x_1, \ldots, x_k$.

Many 3-AP decompositions exist; for example,

$$\mathbb{U}_{31} = \langle 25 \rangle \times \langle 30 \rangle \times \langle 4 \rangle,$$

where the cyclic factors have orders 3, 2, 5 respectively.

Problems

- Do k-AP decompositions of $\mathbb{U}_p$ exist for $k > 3$? (No 4-AP decompositions exist for $p < 10000$.)
- What about composite moduli? (For example,

$$\mathbb{U}_{104} = \langle 77 \rangle \times \langle 79 \rangle \times \langle 81 \rangle \times \langle 83 \rangle,$$

  where the factors have orders 2, 2, 3, 4 respectively.

Problem BCC 22.25 Sumfree subsets of $\mathbb{Z}_p$ Presented by A. Sapozhenko
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Let A be a sum-free subset of $\mathbb{Z}_p$ (that is, if $x, y \in A$, then $x + y \notin A$).

Lev [3] showed that, if $|A| = m \leq 0.33p$, then there exists an element $k \in [1, \ldots, (p-1)/2]$ such that $kA \subseteq [m, \ldots, p-m]$ (where $kA = \{ka : a \in A\}$).

Moreover, Deshouillers and Freiman [1] and Deshouillers and Lev [2] proved that 0.33 can be replaced by 0.324 and 0.318 respectively.

This is known to be false if 0.33 is replaced by 0.25.

Problem Show that the theorem is true if 0.33 is replaced by any smaller number which is greater than 0.25.

References

[1] J.-M. Deshouillers and G. A. Freiman, On sum-free sets modulo p, *Functiones et Approximatio* **35** (2006), 7–15.

[2] J.-M. Deshouillers and V. F. Lev, Refined bound for sum-free sets in groups of prime order, *J. London Math. Soc.* (to appear).

[3] V. F. Lev, Large sum-free sets in Z/Z_p, *Israel J. Math.* **154**(2006), 221–234.

Miscellaneous problems

Problem BCC 22.26 Determinant and permanent Presented by Ian Wanless
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The *permanent* of a square matrix is the sum of all the products occurring in the expansion of the determinant, but without the signs affixed to these terms in the determinant expansion. For example, the permanent of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is 2.

Among the $n \times n$ matrices A of zeros and ones which satisfy $|\det(A)| = \text{per}(A)$, what is the maximum possble value of $|\det(A)|$?

For $n = 1, \ldots, 8$, the maxima are 1, 1, 2, 3, 5, 8, 24, 24. An example of a 7×7 matrix with determinant and permanent 24 is the incidence matrix of the *Fano plane* (the unique Steiner triple system with 7 points). See the paper of Cavenagh and Wanless cited above under Problem BCC22.21 for more discussion.

Problem BCC 22.27 Ramsey Theory Contest Presented by David Cariolaro
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Problem Prove (without using computers) that the Ramsey number $R(3, 7)$ is equal to 23; in other words, any 2-colouring of the edges of the complete graph on 23 vertices has either a triangle of the first colour or a complete subgraph of size 7 of the second (but this is not true for 22 vertices).

Solutions should be sent to the proposer and the editor not later than 1 June 2011; the most elegant solution will be awarded a prize of £600 at the 23rd British Combinatorial Conference.

Some relevant references are appended.

References

[1] G. Kery, Ramsey egy graftelmaleti, *Mat. Lapok* **15** (1964), 204–224 (in Hungarian).

[2] J. G. Kalbfleisch, *Chromatic graphs and Ramsey's theorem*, Ph.D. thesis, University of Waterloo, 1966.

[3] J. E. Graver and J. Yackel, Some graph theoretic results associated with Ramsey's theorem, *J. Combinatorial Theory* **4** (1968), 125–175.

[4] A. Rafiey, On the Ramsey number $R(3,7)$, unpublished, available from `http://tech.groups.yahoo.com/group/Graphs`

[5] D. Cariolaro, On the Ramsey number $R(3,6)$, *Austral. J. Combinatorics* **37** (2007), 301–304.