The power graph of a finite group

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Abstract

The *power graph* of a group is the graph whose vertex set is the group, two elements being adjacent if one is a power of the other. We observe that non-isomorphic finite groups may have isomorphic power graphs, but that finite abelian groups with isomorphic power graphs must be isomorphic. We conjecture that two finite groups with isomorphic power graphs have the same number of elements of each order. We also show that the only finite group whose automorphism group is the same as that of its power graph is the Klein group of order 4.

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1. Introduction

There are a number of constructions of graphs from groups or semigroups in the literature: among these are intersection graphs of subsemigroups or subgroups (Bosák [1], Zelinka [6]), and of course Cayley graphs, which have a long history.

The *directed power graph* of a semigroup S was defined by Kelarev and Quinn [4] as the digraph $\vec{\mathcal{G}}(S)$ with vertex set S, in which there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m. Motivated by this, Chakrabarty *et al.* [2] defined the (undirected) power graph $\mathcal{G}(S)$, in which distinct x and y are joined if one is a power of the other. This paper concerns the general question: *what information does the power graph of S give us about S?*

An element $e \in S$ is called an *idempotent* if $e^2 = e$. If S is finite, then for each $a \in S$, there exists a natural number m such that a^m is an idempotent. Define a binary relation ρ on S by $a\rho b \Leftrightarrow a^m = b^m$ for some natural number m. Clearly ρ is an equivalence relation on S. In [2], it was shown that, for

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any $a \neq b$ in S, there is a path from a to b in $\mathcal{G}(S)$ if and only if $a\rho b$. So every vertex in $\mathcal{G}(S)$ is adjacent to one and only one idempotent in S, and no two idempotents are connected by a path. Thus the components of $\mathcal{G}(S)$ are precisely the equivalence classes of ρ, and each component contains a unique idempotent to which every other vertices of that component are adjacent. In particular, $\mathcal{G}(G)$ is connected for any finite group G. Moreover in [2], it is shown that for a finite group G, $\mathcal{G}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and for some natural number m.

2. The power graph of a group

In this paper we restrict our attention to groups. Two natural questions are:

- Clearly $G_1 \cong G_2$ implies $\mathcal{G}(G_1) \cong \mathcal{G}(G_2)$. Does the converse hold?
- Clearly the automorphism group of G is contained in the automorphism group of $\mathcal{G}(G)$. When does equality hold?

The answer to the first question is negative for infinite abelian groups. Let C_{p^∞} be the group of rational numbers with p-power denominators mod 1, where p is prime. Then $\mathcal{G}(C_{p^\infty})$ is a countably infinite complete graph, independent of the chosen prime.

It is false for finite groups in general. Let G be a finite group of exponent 3, that is, satisfying $x^3 = 1$ for all $x \in G$. Then $\mathcal{G}(G)$ consists of $(|G| - 1)/2$ triangles sharing a common vertex (the identity). The elementary abelian group (the direct product of cyclic groups of order 3) has exponent 3, but there are non-abelian groups as well: the smallest is the group of order 27 with presentation

$$G = \langle x, y \mid x^3 = y^3 = [x, y]^3 = 1\rangle,$$

where $[x, y]$ is the commutator $x^{-1}y^{-1}xy$.

In fact, the phenomenon is not uncommon. A short calculation with GAP [3] and its package GRAPE [5] revealed that there are two pairs of groups of order 16 with isomorphic power graphs, and two pairs of groups of order 27. The phenomenon becomes more common; for order 32 there are one quadruple, two triples, and eight pairs.

However, we have a positive result for finite abelian groups:

Theorem 1. *Let G_1 and G_2 be finite abelian groups with $\mathcal{G}(G_1) \cong \mathcal{G}(G_2)$. Then $G_1 \cong G_2$.*

Proof. In this proof, we assume that G is a finite abelian group, and $\Gamma = \mathcal{G}(G)$. If G is a cyclic group of prime power order, then Γ is a complete graph (and conversely); so we may assume that this is not the case.

Define a relation $\equiv$ on G by the rule that $x \equiv y$ if $\{x\} \cup \Gamma(x) = \{y\} \cup \Gamma(y)$, where $\Gamma(x)$ is the set of neighbours of x in Γ. It is clear that, if $\langle x\rangle = \langle y\rangle$ (that is, x and y generate the same cyclic subgroup), then each is a power of the other, and so $x \equiv y$. Our first task is to prove that the converse is true.

So suppose that $x \equiv y$. Then one of x and y is a power of the other, say $y = x^k$. Replacing x by another generator of the same cyclic subgroup if necessary, we may assume that k divides the order of x; say $o(y) = m$ and $o(x) = km$. Assume for a contradiction that $k > 1$. Every element in the cyclic group generated by x is joined to x, and hence also to y. This subgroup contains elements of all orders dividing mk, so we conclude that, if d divides mk, then either d divides k or k divides d. Clearly this implies that m and k are powers of the same prime.

Since, by assumption, G is not cyclic of prime power order, there is an element u of prime order q not contained in $\langle x \rangle$. It is straightforward to construct an element z such that y but not x is a power of z. (If $q = p$, take $z = xu$; otherwise take $z = yu$.) This yields the desired contradiction.

Now we can prove the result in the case that G is a p-group for some prime p. In this case, the equivalence class of an element of order p^i has size $p^{i-1}(p-1)$, so we can reconstruct the orders of all elements. It is well known that this information determines the isomorphism type of G.

Now we do the general case. Let p be the smallest prime divisor of $|G|$. Then the elements of order p are those lying in equivalence classes of size $p-1$. Inductively, elements of order p^i are those lying in equivalence classes of size $p^{i-1}(p-1)$ with the property that the only smaller classes to which they are adjacent are elements already found. So we can construct the Sylow p-subgroup P of G, and recognize it by the numbers of elements of each possible order it contains.

Now an element has order coprime to p if and only if it is not joined to any element of P except the identity. So we can find the complement H to P in G, and inductively we can recognize its structure. Then $G \cong P \times H$, so we know the structure of G.

In view of the fact that Theorem 1 fails for arbitrary finite groups, we replace the first question with another:

Conjecture 1. Let G_1 and G_2 be finite groups with $\mathcal{G}(G_1) \cong \mathcal{G}(G_2)$. Then G_1 and G_2 have the same numbers of elements of each order.

This has been tested for groups of order up to 127 by computer; it is true in all these cases. We note that the converse fails for groups of order 16. In addition, we have the following result:

Proposition 2. *Let G_1 and G_2 be finite groups with $\vec{\mathcal{G}}(G_1) \cong \vec{\mathcal{G}}(G_2)$. Then G_1 and G_2 have the same numbers of elements of each order.*

Proof. In the directed power graph of G, the maximal complete digraphs (sets of vertices joined by arcs in both directions) are clearly the equivalence classes of the relation $\equiv$, where $x \equiv y$ if $\langle x \rangle = \langle y \rangle$. The equivalence classes are partially ordered by the relation $[y] \leq [x]$ if there is an arc from x to y. A minimal non-identity equivalence class consists of elements of prime order p, and has cardinality $p-1$. For any equivalence class $[x]$, we can now determine the set of primes dividing the order of x from the set of minimal classes below $[x]$ in the partial order. If x has order $n = p_1^{a_1} \cdots p_r^{a_r}$, we now know the primes $p_1, \ldots, p_r$ and the value of $\phi(n) = |[x]|$. The exponent a_i is one more than the exponent of the power of p_i dividing $|[x]|$. So the order of x is determined.

For the second question, we have a complete answer for finite groups.

Theorem 3. *The only finite group G for which* $\mathrm{Aut}(G) = \mathrm{Aut}(\mathcal{G}(G))$ *is the Klein group $C_2 \times C_2$.*

Proof. Let G satisfy the condition of the proposition. First note that the map $x \mapsto x^{-1}$ is always an automorphism of $\mathcal{G}(G)$, but is an automorphism of G if and only if G is abelian. So our group G is abelian.

If x is an element of order greater than 2, and y an non-identity element not in $\langle x \rangle$ whose order divides that of x, then there is a graph automorphism fixing x and y and mapping xy to its inverse; no group automorphism can do this. We conclude that G is either cyclic or an elementary abelian 2-group.

Suppose that G is cyclic of order m. If m is composite and greater than 4, let k be a divisor of m such that $m/k > 2$. There is a graph automorphism fixing the generator x of G and mapping x^k to its inverse, which is impossible for a group automorphism. If m is a prime power, then $\mathcal{G}(G)$ is a complete graph and has an automorphism moving the identity, which no group automorphism can do.

Let G be an elementary abelian 2-group of order 2^a. Then $\mathcal{G}$ is a star $K_{1,2^a-1}$, then $\mathrm{Aut}(\mathcal{G}(G))$ is the symmetric group of degree $2^a - 1$, while $\mathrm{Aut}(G)$ is the general linear group $\mathrm{GL}(a, 2)$; these groups are equal if and only if $a = 2$.

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