

Asymptotic enumeration of 2-covers and line graphs

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Abstract

A *2-cover* is a multiset of subsets of $[n] := \{1, 2, \dots, n\}$ such that each element of $[n]$ lies in exactly two of the subsets. A 2-cover is called *proper* if all of the subsets are distinct, and is called *restricted* if any two of them intersect in at most one element.

In this paper we find asymptotic enumerations for the number of line graphs on n -labelled vertices and for 2-covers.

We find that the number s_n of 2-covers and the number t_n of proper 2-covers both have asymptotic growth

$$s_n \sim t_n \sim B_{2n} 2^{-n} \exp\left(-\frac{1}{2} \log(2n/\log n)\right),$$

where B_{2n} is the $2n$ th Bell number. Moreover, the numbers u_n of restricted 2-covers on $[n]$ and v_n of restricted, proper 2-covers on $[n]$ and l_n of line graphs all have growth

$$u_n \sim v_n \sim l_n \sim B_{2n} 2^{-n} n^{-1/2} \exp\left(-\left[\frac{1}{2} \log(2n/\log n)\right]^2\right).$$

KEYWORDS: ASYMPTOTIC ENUMERATION, LINE GRAPHS, SET PARTITIONS

1 Introduction

A 2-cover of $[n] := \{1, 2, \dots, n\}$ is a multiset of subsets $\{S_1, S_2, \dots, S_m\}$, $S_i \subseteq [n]$, (possibly with $S_i = S_j$ for some $i \neq j$), such that for each $d \in [n]$

the number of j such that $d \in S_j$ is exactly 2. A 2-cover is called *proper* if $S_i \neq S_j$ whenever $i \neq j$. A 2-cover is called *restricted* if the intersection of any 2 of the S_i contains at most one element. These definitions have been taken from [4]. Note that a proper 2-cover $\{S_1, \dots, S_m\}$ is a set.

The *line graph* $L(G)$ of a simple graph G is the graph whose vertex set is the edge set of G and such that two vertices are adjacent in $L(G)$ if and only if the corresponding edges of G have a common vertex.

Let s_n be the number of 2-covers of $[n]$; let t_n be the number of proper 2-covers of $[n]$; let u_n be the number of restricted 2-covers of $[n]$; let v_n be the number of restricted, proper 2-covers of $[n]$; and let l_n be the number of line graphs on n labelled vertices. Let B_n be the n th Bell number. Given sequences a_n and b_n , we write $a_n \sim b_n$ to mean $\lim_{n \rightarrow \infty} a_n/b_n = 1$.

Theorem 1 *The number of 2-covers and the number of proper 2-covers have asymptotic growth*

$$s_n \sim t_n \sim B_{2n} 2^{-n} \exp\left(-\frac{1}{2} \log(2n/\log n)\right) = B_{2n} 2^{-n} \sqrt{\frac{\log n}{2n}} \quad (1)$$

while the number of restricted 2-covers, restricted, proper 2-covers and line graphs all have asymptotic growth

$$u_n \sim v_n \sim l_n \sim B_{2n} 2^{-n} n^{-1/2} \exp\left(-\left[\frac{1}{2} \log(2n/\log n)\right]^2\right). \quad (2)$$

We note that the expression in the exponential in (2) is the square of the expression in the exponential in (1).

The main term $B_{2n} 2^{-n}$ in (1) and (2) can be roughly explained as follows. Take $2n$ half edges $\{1, 2, \dots, 2n\}$, partition them into blocks, and form n edges $\{j, j+n\}$ for $j \in \{1, 2, \dots, n\}$, making sure j and $j+n$ go into different blocks for all j to avoid loops.

We make some initial observations regarding 2-covers, special graphs and orbits in Section 2. We use a probabilistic method to prove (1) in Section 3. A pair of technical lemmas are proven in Section 3.1, (1) is proven for s_n in Section 3.2 and it is proven for t_n in Section 3.3. We prove (2) in Section 4.

In both probabilistic and generating function proofs we will make use of Lambert's W -function $W(t)$, which is a solution to

$$W(t)e^{W(t)} = t \quad (3)$$

and which has asymptotics (see (3.10) of [7])

$$W(t) = \log t - \log \log t + \frac{\log \log t}{\log t} + o\left(\frac{1}{\log t}\right) \quad \text{as } t \rightarrow \infty. \quad (4)$$

For each k -cover $\{S_1, \dots, S_m\}$ of $[n]$ we define an associated $m \times n$ incidence matrix M with entries given by

$$M_{i,j} = \begin{cases} 1 & \text{if } j \in S_i; \\ 0 & \text{if } j \notin S_i. \end{cases}$$

Note that M has exactly k ones in each column and that the ordering of the rows is arbitrary. A k -cover is proper if and only if M has no repeated rows. A k -cover is restricted if and only if M has no repeated columns. Therefore, Theorem 1 is equivalent to the asymptotic enumeration of certain 0-1 matrices. The general methods of this paper were used for the asymptotic enumeration of other 0-1 matrices called incidence matrices in [2, 3].

2 2-covers, line graphs and orbits

In this section we establish correspondences between 2-covers, line graphs and orbits of certain permutation groups.

2.1 2-covers and graphs

We define a *special multigraph* to be a multigraph with no isolated vertices or loops. Our first result is

Proposition 1 *There is a bijection between 2-covers on $[n]$ and special multigraphs having unlabelled vertices and n labelled edges, such that*

- *proper 2-covers correspond to multigraphs having no connected component of size 2;*
- *restricted 2-covers correspond to simple graphs.*

Proof Let $\{S_1, \dots, S_m\}$ be a 2-cover of $[n]$. Construct a graph G as follows:

- the vertex set is $[m]$;

- for each $i \in [n]$, there is an edge e_i joining vertices j and k , where S_j and S_k are the two sets of the 2-cover containing i .

The graph G is a multigraph (that is, repeated edges are permitted), but it has no isolated vertices and no loops.

Conversely, given a multigraph without isolated vertices or loops, we can recover a 2-cover: number the edges e_1, \dots, e_n , and let S_i be the set of indices j for which the i th vertex lies on edge e_j . Thus we have the first part of the proposition.

The second part comes from observing that a “repeated set” in a 2-cover corresponds to a pair of vertices lying on the same edges, while a pair of elements lying in two different sets correspond to a pair of edges incident to the same two vertices. ■

2.2 Generating function identities for 2-covers

Recall that s_n , t_n , u_n and v_n denote the numbers of 2-covers, proper 2-covers, restricted 2-covers, and restricted proper 2-covers respectively. Using Proposition 1 in this subsection we will find relationships between these quantities and derive corresponding generating function identities.

Proposition 2 *Let $S(n, k)$ denote the Stirling numbers of the second kind, that is, the number of set partitions of $[n]$ into exactly k nonempty subsets. Then,*

$$\begin{aligned} s_n &= \sum_{k=1}^n S(n, k) u_k \\ t_n &= \sum_{k=1}^n S(n, k) v_k \\ u_n &= \sum_{k=0}^n \binom{n}{k} v_k \end{aligned}$$

Proof We prove these for the corresponding special multigraphs.

Any special multigraph with edges e_1, \dots, e_n can be described by giving a partition of $[n]$ into, say, k parts, together with a special simple graph with k labelled edges; simply replace the i th edge of the simple graph by the i th set of edges of the partition (where the edges are ordered lexicographically, say).

This is clearly a bijection. Moreover, the simple graph has no connected components of size 2 if and only if the same holds for the multigraph. This proves the first two equations.

Given a special simple graph, there is a distinguished subset of $[n]$ (of size $n - k$, say) consisting of isolated edges; the remaining graph has no components of size 2. Again, the correspondence is bijective. So the third equation holds. ■

Proposition 2 can be reformulated in terms of exponential generating functions. Let $S(x) = \sum_{n \geq 0} s_n x^n / n!$, with similar definitions for the others. The proof of Proposition 3 is omitted.

Proposition 3

$$\begin{aligned} S(x) &= U(e^x - 1) \\ T(x) &= V(e^x - 1) \\ U(x) &= V(x)e^x. \end{aligned}$$

It follows from Proposition 3 that $S(x) = T(x)B(x)$, where $B(x) = e^{e^x - 1}$ is the exponential generating function for the Bell numbers. This is easily proved directly.

2.3 Unrestricted 2-covers and orbits

A permutation group G acting on an infinite set Ω is *oligomorphic* if it has only finitely many orbits on the set of n -tuples of distinct elements of Ω (equivalently, on the set of all n -tuples). We denote the numbers of these orbits by $F_n(G)$ and $F_n^*(G)$ respectively.

By [6], if G is oligomorphic and *primitive* (that is, preserves no non-trivial equivalence relation on Ω), then $F_n(G) = c^n n! / p(n)$, where $c > 1$ is an absolute constant and $p(n)$ is a polynomial. There is some interest in groups G with the growth of $F_n(G)$ close to this bound. One example is the permutation group $S_\infty^{\{2\}}$ induced by the infinite symmetric group on the set of 2-element subsets of its domain.

Proposition 4 $F_n(S_\infty^{\{2\}}) = u_n$ and $F_n^*(S_\infty^{\{2\}}) = s_n$.

Proof Simply observe that an n -tuple of distinct 2-sets is the edge set of a special simple graph with n labelled edges, while an arbitrary n -tuple of 2-sets is the edge set of a special multigraph with n labelled edges. ■

We note that the relation

$$F_n^*(G) = \sum_{k=1}^n S(n, k) F_k(G)$$

gives an alternative proof of the first equation in Proposition 2. We do not know of a similar interpretation of the other two parameters.

2.4 Generating function identities for line graphs

The relationship between line graphs and 2-covers is contained in Proposition 1. Let $L(x) = \sum_{n \geq 0} l_n x^n / n!$. We now prove

Proposition 5

$$L(x) = e^{-x^3/3! - 6x^4/4! - 15x^5/5! - 15x^6/6!} U(x).$$

Proof According to Whitney's Theorem [5], an isomorphism between line graphs $L(G_1)$ and $L(G_2)$ of connected graphs is induced by an isomorphism from G_1 to G_2 , except in one case: the line graphs of the triangle K_3 and the star $K_{1,3}$ are isomorphic. Moreover, Sabidussi [9] has shown that if G is a connected graph with at least three vertices, then the automorphism groups of G and $L(G)$ are isomorphic if G is not K_4 , K_4 with an edge deleted or K_4 with two adjacent edges deleted, which we shall denote by K'_4 and K''_4 , respectively.

Now the connected components of line graphs which are triangles contribute a factor $e^{x^3/3!}$ to the exponential generating function $L(x)$ for line graphs on $[n]$; that is, $L(x) = e^{x^3/3!} W'(x)$, where $W'(x)$ is the e.g.f. for line graphs with no such components. Similarly, components which are triangles or stars contribute a factor $(e^{x^3/3!})^2$ to the e.g.f. for special simple graphs with n edges, leading to an overall multiplication by a factor of $e^{-x^3/3!}$

Next, while K_4 has S_4 as an automorphism group and therefore admits $6!/4! = 30$ different edge labellings, the order of the automorphism group of $L(K_4)$ is $2 \cdot 4!$ and therefore $L(K_4)$ admits 15 different vertex labellings. Similar to above, this leads to a correction by a factor of $e^{-15x^6/6!}$.

Similar arguments hold for K'_4 and K''_4 , leading to factors $e^{-15x^5/5!}$ and $e^{-6x^4/4!}$, correspondingly.

Proposition 5 now follows by Whitney's Theorem, Sabidussi's result, and Proposition 3. ■

3 Unrestricted 2-covers: a probabilistic approach

In this section we prove (1) of Theorem 1 by using a probabilistic construction.

3.1 Technical results

We proceed with the following definitions and lemma. Let T_n be the set of proper 2-covers on $[n]$. Let \mathcal{S}_n be the set of set partitions of $[2n]$. Let $E_{1,n} \subset \mathcal{S}_n$ be the subset of set partitions of $[2n]$ such that j and $j+n$ are contained in different blocks for each $j \in [n]$. Define the function ψ from a subset \tilde{S} of $[2n]$ to a subset of $[n]$ by $\psi(\tilde{S}) = \{j : j \in \tilde{S} \text{ or } j+n \in \tilde{S}\}$. Let $E_{2,n} \subset \mathcal{S}_n$ be the subset of set partitions of $[2n]$ with blocks $\{\tilde{S}_1, \dots, \tilde{S}_m\}$ such that $\psi(\tilde{S}_{i_1}) \neq \psi(\tilde{S}_{i_2})$ for each $i_1 \neq i_2$. Let $C_n = E_{1,n} \cap E_{2,n}$. Let ϕ be the function on \mathcal{S}_n given by

$$\phi(\{\tilde{S}_1, \dots, \tilde{S}_m\}) = \{\psi(\tilde{S}_1), \dots, \psi(\tilde{S}_m)\}.$$

Lemma 1 ϕ maps C_n onto T_n and $|\phi^{-1}(\mathbf{a})| = 2^n$ for all $\mathbf{a} \in T_n$.

Proof Fix $\{\tilde{S}_1, \dots, \tilde{S}_m\} \in C_n$. Each $j \in [n]$ appears in exactly two blocks of $\phi(\{\tilde{S}_1, \dots, \tilde{S}_m\})$ because of the definition of $E_{1,n}$ and the blocks of $\{\tilde{S}_1, \dots, \tilde{S}_m\}$ are unique because of the definition of $E_{2,n}$ so $\phi(\{\tilde{S}_1, \dots, \tilde{S}_m\}) \in T_n$.

Let $\mathbf{a} = \{S_1, \dots, S_m\} \in T_n$. For each $j \in [n]$ there are two ways of assigning $j, j+n$ to the appearances of j in \mathbf{a} (think of a fixed ordering of the blocks of \mathbf{a} to see this). The choices made for every $j \in [n]$ determine an *assignment*. Clearly, every element of $\phi^{-1}(\mathbf{a})$ must be of the form $\chi(\mathbf{a})$ for some assignment χ . There are 2^n assignments. We also write $\chi(S_i)$ for the block \tilde{S}_i corresponding to S_i in $\chi(\mathbf{a})$.

We claim that each assignment $\chi(\mathbf{a})$ gives a unique element of C_n . To see this, first note that j and $j+n$ are clearly in different blocks of $\chi(\mathbf{a})$, so $\chi(\mathbf{a}) \in E_{1,n}$. Secondly, $\phi \circ \chi$ is the identity map on T_n . Therefore, $\chi(\mathbf{a}) \in E_{2,n}$ because \mathbf{a} is a proper 2-cover. Moreover, $\chi_1(\mathbf{a}_1) \neq \chi_2(\mathbf{a}_2)$ for all $\mathbf{a}_1, \mathbf{a}_2 \in T_n$ such that $\mathbf{a}_1 \neq \mathbf{a}_2$ and for all assignments χ_1 and χ_2 , which gives $\phi^{-1}(\mathbf{a}_1) \cap \phi^{-1}(\mathbf{a}_2) = \emptyset$.

We next prove that if χ_1 and χ_2 are two assignments such that $\chi_1(\mathbf{a}) = \chi_2(\mathbf{a})$, then $\chi_1 = \chi_2$. To see this, let

$$\mathcal{U} = \{j \in [n] : \chi_1 \text{ and } \chi_2 \text{ differ for } j\}.$$

Without loss of generality, assume that $j \in S_1$ and $j \in S_2$. Then, either $j \in \chi_1(S_1)$ and $j \in \chi_2(S_2)$ or $j + n \in \chi_1(S_1)$ and $j + n \in \chi_2(S_2)$. It follows that $\chi_1(S_1) = \chi_2(S_2)$. Therefore, $\phi \circ \chi_1(S_1) = \phi \circ \chi_2(S_2)$ or $S_1 = S_2$ violating the assumption that \mathbf{a} is proper. We conclude that $\mathcal{U} = \emptyset$ and that $\chi_1 = \chi_2$. This implies that $|\phi^{-1}(\mathbf{a})| = 2^n$. ■

Next we generalize Lemma 1 to (possibly) improper covers. Let U_n denote the set of 2-covers of $[n]$.

Lemma 2 *ϕ maps $E_{1,n}$ onto U_n . Let $\mathbf{a} = \{S_1, S_2, \dots, S_m\}$ be a 2-cover of $[n]$. Let \mathcal{M} be the set of $i \in [m]$ such that there does not exist any $j \in [m] \setminus \{i\}$, $S_j = S_i$. Let*

$$\rho = \frac{m - |\mathcal{M}|}{2}$$

be the number of pairs $\{i, j\}$ such that $S_i = S_j$. Then

$$|\phi^{-1}(\mathbf{a})| = 2^{n-\rho}.$$

Proof Clearly ϕ maps $E_{1,n}$ onto U_n . Let $\mathcal{N} = [n] \setminus \{\cup_{i \in \mathcal{M}} S_i\}$. Then $\{S_i : i \in \mathcal{M}\}$ is a proper cover of \mathcal{N} and Lemma 1 implies that

$$|\phi^{-1}(\{S_i : i \in \mathcal{N}\})| = 2^{|\mathcal{N}|}.$$

For each pair S_{i_1}, S_{i_2} such that $i_1 \neq i_2$ and $S_{i_1} = S_{i_2}$, it must be true that $\phi^{-1}(S_i)$ consists of two sets \tilde{S}_1 and \tilde{S}_2 such that for each $j \in S_{i_1}$ either $j \in \tilde{S}_1$ and $j + n \in \tilde{S}_2$ or $j + n \in \tilde{S}_1$ and $j \in \tilde{S}_2$. The number of unordered sets $\tilde{S}_{i_1}, \tilde{S}_{i_2}$ is $2^{|S_{i_1}|-1}$. Therefore,

$$|\phi^{-1}(\mathbf{a})| = 2^{|\mathcal{N}|} \prod 2^{|S_{i_1}|-1} = 2^{n-\rho},$$

where the product is over pairs i_1, i_2 such that $i_1 \neq i_2$ and $S_{i_1} = S_{i_2}$. ■

3.2 Asymptotic enumeration of proper 2-covers

From Lemma 1 we conclude that $|C_n| = 2^n t_n$ so

$$t_n = 2^{-n} |C_n| = 2^{-n} \frac{|C_n|}{B_{2n}} B_{2n} \tag{5}$$

where B_{2n} is the $2n$ th Bell number.

We will now prove

Lemma 3

$$\frac{|E_{1,n}|}{B_{2n}} \sim \sqrt{\frac{\log n}{2n}} \quad (6)$$

and

$$\frac{|E_{2,n}|}{B_{2n}} = 1 - O\left(\frac{\log^2 n}{n}\right). \quad (7)$$

Proof To prove (6), choose an element of \mathcal{S}_n uniformly at random and let X be the number of $j \in [n]$ for which j and $j+n$ are in the same block. We have

$$\mathbb{P}(X = 0) = \frac{|E_{1,n}|}{B_{2n}}. \quad (8)$$

We have $X = \sum_{j=1}^n I_j$ where I_j is the indicator random variable that j and $j+n$ are in the same block. The r th falling moment of X is

$$\begin{aligned} \mathbb{E}(X)_r &= \mathbb{E}X(X-1)\cdots(X-r+1) \\ &= \sum \mathbb{E}(I_{j_1}I_{j_2}\cdots I_{j_r}) \end{aligned}$$

where the sum is over (j_1, \dots, j_r) with no repetitions. To find $\mathbb{E}(I_{j_1}I_{j_2}\cdots I_{j_r})$ we take $[2n] \setminus \{j_1, j_2, \dots, j_r\}$ and form a set partition. We then add j_k to the block containing j_k+n for each $k \in [r]$. This process is uniquely reversible. Therefore,

$$\mathbb{E}(X)_r = \frac{(n)_r B_{2n-r}}{B_{2n}}.$$

We apply the formula in Corollary 13, page 18, of [1] to obtain

$$\mathbb{P}(X = 0) = \sum_{r=0}^{\infty} (-1)^r \frac{\mathbb{E}(X)_r}{r!} = \sum_{r=0}^{\infty} \frac{(-1)^r (n)_r B_{2n-r}}{r! B_{2n}}. \quad (9)$$

To analyze (9) we use the expansion of the Bell numbers [7, 10]

$$\begin{aligned} \log B_n &= e^w(w^2 - w + 1) - \frac{1}{2} \log(1+w) - 1 - \frac{w(2w^2 + 7w + 10)}{24(1+w)^3} e^{-w} \\ &\quad - \frac{w(2w^4 + 12w^3 + 29w^2 + 40w + 36)}{48(1+w)^6} e^{-2w} + O(e^{-3w}), \end{aligned}$$

where $w = W(n)$ is given by (3), (4), from which we obtain (using Maple)

$$\log B_{n-r} - \log B_n = -rw + \frac{rw}{2n} \left(\frac{r}{w+1} + \frac{1}{(w+1)^2} \right) + O\left(\frac{r^3 w}{n^2}\right).$$

In particular,

$$\frac{B_{n-1}}{B_n} \sim \frac{\log n}{n}$$

so there exists a constant $C > 0$ such that

$$\frac{B_{n-r}}{B_n} \leq \frac{(C \log n)^r}{(n)_r}. \quad (10)$$

Moreover,

$$\begin{aligned} \log B_{2n-r} - \log B_{2n} &= -rv + \frac{rv}{4n} \left(\frac{r}{v+1} + \frac{1}{(v+1)^2} \right) + O\left(\frac{r^3 v}{n^2}\right) \\ &= -r \log n + rc_n + r^2 d_n + O\left(\frac{r^3 \log n}{n^2}\right), \end{aligned}$$

where $v = W(2n)$ has the expansion

$$v = \log n - \log \log n + \log 2 + \frac{\log \log n}{\log n} - \frac{\log 2}{\log n} + o\left(\frac{1}{\log n}\right),$$

where

$$\begin{aligned} c_n &= \log n - v - \frac{rv}{4n(v+1)^2} \\ &= \log \log n - \log 2 - \frac{\log \log n}{\log n} + \frac{\log 2}{\log n} + o\left(\frac{1}{\log n}\right) \end{aligned}$$

and where

$$d_n = O\left(\frac{1}{n}\right).$$

Using (10) we estimate

$$\begin{aligned}
\left| \sum_{r > \log^{3/2} n} (-1)^r \frac{\mathbb{E}(X)_r}{r!} \right| &\leq \sum_{r > \log^{3/2} n} \frac{(n)_r B_{2n-r}}{r! B_{2n}} \\
&\leq \sum_{r > \log^{3/2} n} \frac{(C \log 2n)^r}{r!} \\
&= (2n)^C \sum_{r > \log^{3/2} n} e^{-C \log 2n} \frac{(C \log 2n)^r}{r!} \\
&= o(1). \tag{11}
\end{aligned}$$

For $r \leq \log^{3/2} n$, we have

$$\frac{B_{2n-r}}{B_{2n}} = n^{-r} \exp \left(r c_n + r^2 d_n + O \left(\frac{\log^9 n}{n^2} \right) \right)$$

and

$$(n)_r = n^r \exp \left(O \left(\frac{r^2}{n} \right) \right),$$

hence

$$\mathbb{E}(X)_r = \exp \left(r c_n + r^2 d_n + O \left(\frac{\log^9 n}{n^2} \right) \right).$$

Therefore,

$$\begin{aligned}
\sum_{0 \leq r \leq \log^{3/2} n} (-1)^r \frac{\mathbb{E}(X)_r}{r!} &= \sum_{0 \leq r \leq \log^{3/2} n} \frac{(-1)^r}{r!} e^{r c_n + r^2 d_n} \left(1 + O \left(\frac{\log^9 n}{n^2} \right) \right) \\
&= \sum_{0 \leq r \leq \log^{3/2} n} \frac{(-1)^r}{r!} e^{r c_n} \left(1 + d_n r^2 + O \left(\frac{\log^9 n}{n^2} \right) \right) \\
&= \sum_{0 \leq r \leq \log^{3/2} n} \frac{(-1)^r}{r!} e^{r c_n} + d_n \sum_{0 \leq r \leq \log^{3/2} n} \frac{(-1)^r r^2}{r!} e^{r c_n} \\
&\quad + O \left(\frac{\log^9 n}{n^2} \right) \sum_{0 \leq r \leq \log^{3/2} n} \frac{e^{r c_n}}{r!}. \tag{12}
\end{aligned}$$

We proceed to approximate the terms in (12). First, we find that

$$\begin{aligned}
\sum_{0 \leq r \leq \log^{3/2} n} \frac{(-1)^r}{r!} e^{rc_n} &= \exp(-e^{c_n}) + O\left(\sum_{\log^{3/2} n \leq r \leq n} \frac{e^{rc_n}}{r!}\right) \\
&= \exp\left(-\frac{\log n}{2} \left[1 - \frac{\log \log n}{\log n} + \frac{\log 2}{\log n} + o\left(\frac{1}{\log n}\right)\right]\right) + o(n^{-1/2}) \\
&\sim \sqrt{\frac{\log n}{2n}}.
\end{aligned} \tag{13}$$

We estimate

$$\begin{aligned}
&d_n \left| \sum_{0 \leq r \leq \log^{3/2} n} \frac{(-1)^r}{r!} r^2 e^{rc_n} \right| \\
&= d_n \left| \sum_{2 \leq r \leq \log^{3/2} n} \frac{(-1)^r}{(r-2)!} e^{rc_n} + \sum_{1 \leq r \leq \log^{3/2} n} \frac{(-1)^r}{(r-1)!} e^{rc_n} \right| \\
&= d_n \left| e^{2c_n} \sum_{2 \leq r \leq \log^{3/2} n} \frac{(-1)^r}{(r-2)!} e^{(r-2)c_n} + e^{c_n} \sum_{1 \leq r \leq \log^{3/2} n} \frac{(-1)^r}{(r-1)!} e^{(r-1)c_n} \right| \\
&= d_n \left(\exp(-e^{c_n} + 2c_n) + \exp(-e^{c_n} + c_n) + O\left(e^{2c_n} \sum_{\log^{3/2} n \leq r \leq n} \frac{e^{rc_n}}{r!}\right) \right) \\
&= o(n^{-1/2}).
\end{aligned} \tag{14}$$

Finally, we have

$$\begin{aligned}
O\left(\frac{\log^9 n}{n^2}\right) \sum_{0 \leq r \leq \log^{3/2} n} \frac{e^{rc_n}}{r!} &\leq O\left(\frac{\log^9 n}{n^2}\right) e^{c_n} \\
&= o(n^{-1/2}).
\end{aligned} \tag{15}$$

Together, (8), (9), (11), (12), (13), (14) and (15) prove (6).

To show (7), let Y be the number of pairs S_i, S_j in an partition in \mathcal{S}_n chosen uniformly at random for which $\psi(S_i) = \psi(S_j)$. For such S_i, S_j of size $|S_i| = |S_j| = k$, the probability that they are present in the random partition is $B(2n - 2k)/B(2n)$. The total number of pairs S_i, S_j of size k is bounded

by $\binom{n}{k}2^k$ (the number of ways of choosing a subset J of size k from $[n]$ times a bound on the number of ways of choosing two subsets S_1, S_2 of $[2n]$ of size k such that either $j \in S_1$ and $j+n \in S_2$ or $j+n \in S_1$ and $j \in S_2$ for all $j \in J$.) Therefore, using (10) we get

$$\begin{aligned}
1 - \frac{|E_{2,n}|}{B_{2n}} &= \mathbb{P}(Y > 0) \\
&\leq \mathbb{E}Y \\
&\leq \sum_{k=1}^n \binom{n}{k} 2^k \frac{B_{2n-2k}}{B_{2n}} \\
&\leq \sum_{k=1}^n \binom{n}{k} 2^k \frac{(C \log 2n)^{2k}}{(2n)_{2k}} \\
&\leq \sum_{k=1}^n \frac{(n)_k (2C^2 \log^2 2n)^k}{(2n)_{2k} k!} \\
&= O\left(\frac{\log^2 n}{n}\right).
\end{aligned}$$

■

Lemma 3 and (5) along with

$$\frac{|C_n|}{B_{2n}} \leq \frac{|E_{1,n}|}{B_{2n}}$$

and

$$\frac{|C_n|}{B_{2n}} \geq \frac{|E_{1,n}| - (B_{2n} - |E_{2,n}|)}{B_{2n}}$$

prove (1) for t_n .

3.3 Asymptotic enumeration of 2-covers

In this subsection we prove (1) for s_n . Recall that U_n denotes the set of 2-covers of $[n]$. Each element of $E_{1,n}$ is mapped to a unique $\mathbf{a} \in U_n$ by ϕ . Given $\omega = \{\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_m\} \in \mathcal{S}_n$, let $Z(\omega)$ be the number of pairs $\{i_1, i_2\}$ such that $\psi(\tilde{S}_{i_1}) = \psi(\tilde{S}_{i_2})$. Note that in the case $\omega \in E_{1,n}$ we have $Z(\omega) = \rho$ with ρ defined with respect to $\mathbf{a} = \phi(\omega)$ in the statement of Lemma 2.

Define $D_{\rho,n}$ for $\rho \in \{0, 1, \dots, n\}$ to be

$$D_{\rho,n} = \{\omega \in E_{1,n} : Z(\omega) = \rho\}.$$

Note that $D_{0,n} = C_n$. By Lemma 2,

$$\begin{aligned}
u_n &= \sum_{\rho=0}^n |D_{\rho,n}| 2^{-n+\rho} \\
&= |C_n| 2^{-n} + \sum_{\rho=1}^n |D_{\rho,n}| 2^\rho \\
&= B_{2n} 2^{-n} \left(\frac{|C_n|}{B_{2n}} + \sum_{\rho=1}^n \frac{|D_{\rho,n}|}{B_{2n}} 2^\rho \right).
\end{aligned}$$

We have shown in the previous section that $C_n/B_{2n} \sim \sqrt{\log n/2n}$. Observe that $\sum_{\rho=1}^n |D_{\rho,n}| 2^\rho / B_{2n} \leq \sum_{\rho=1}^n \mathbb{P}(Z = \rho) 2^\rho$, where Z was defined in the last paragraph and ω is chosen uniformly at random from \mathcal{S}_n . In light of these observations, to prove (1) for s_n it suffices to prove that

$$\sum_{\rho=1}^n \mathbb{P}(Z = \rho) 2^\rho = o\left(\sqrt{\frac{\log n}{2n}}\right). \quad (16)$$

The quantity $\mathbb{P}(Z \geq \rho)$ is equal to the probability that the randomly chosen element of \mathcal{S}_n contains at least ρ disjoint pairs of equal sets, therefore,

$$\mathbb{P}(Z \geq \rho) \leq \sum_{s_1=1}^n \sum_{s_2=1}^n \cdots \sum_{s_\rho=1}^n \binom{n}{s_1, s_2, \dots, s_\rho, n - \sum s_i} \frac{B_{2n-2\sum s_i}}{B_{2n}}$$

Let σ be defined by $\sigma = \sum_{i=1}^\rho s_i$. We can assume $\sigma \leq n$. From (10) we have

$$\begin{aligned}
\mathbb{P}(Z \geq \rho) &\leq \sum_{s_1=1}^n \sum_{s_2=1}^n \cdots \sum_{s_\rho=1}^n \binom{n}{s_1, s_2, \dots, s_\rho, n - \sigma} \frac{(C \log n)^{2\sigma}}{(2n)_{2\sigma}} \\
&= \sum_{s_1=1}^n \sum_{s_2=1}^n \cdots \sum_{s_\rho=1}^n \frac{(n)_\sigma}{\prod_i s_i!} \frac{(C \log n)^{2\sigma}}{(2n)_{2\sigma}}.
\end{aligned}$$

Observing that

$$\frac{(n)_\sigma}{(2n)_{2\sigma}} = \frac{(n)_\sigma}{(2n)_\sigma (2n - \sigma)_\sigma} \leq \frac{1}{(2n)_\sigma} \leq n^{-\sigma},$$

we have

$$\begin{aligned}\mathbb{P}(Z \geq \rho) &\leq \sum_{\sigma=\rho}^n \sum_{\substack{s_1, \dots, s_\rho: \\ \sum_i s_i = \sigma}} \frac{1}{\prod_i s_i!} \left(\frac{C^2 \log^2 n}{n} \right)^\sigma \\ &= \sum_{\sigma=\rho}^n \frac{\rho^\sigma}{\sigma!} \left(\frac{C^2 \log^2 n}{n} \right)^\sigma\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\rho=1}^n \mathbb{P}(Z = \rho) 2^\rho &\leq \sum_{\rho=1}^n \mathbb{P}(Z \geq \rho) 2^\rho \\ &\leq \sum_{\rho=1}^n \sum_{\sigma=\rho}^n \frac{2^\rho \rho^\sigma}{\sigma!} \left(\frac{C^2 \log^2 n}{n} \right)^\sigma \\ &= \sum_{\sigma=1}^n \sum_{\rho=1}^{\sigma} \frac{2^\rho \rho^\sigma}{\sigma!} \left(\frac{C^2 \log^2 n}{n} \right)^\sigma \\ &\leq \sum_{\sigma=1}^n \sum_{\rho=1}^{\sigma} \frac{\rho^\sigma}{\sigma!} \left(\frac{2C^2 \log^2 n}{n} \right)^\sigma \\ &\leq \sum_{\sigma=1}^n \frac{(\sigma+1)^\sigma}{\sigma!} \left(\frac{2C^2 \log^2 n}{n} \right)^\sigma \\ &= O\left(\frac{\log^2 n}{n}\right) \\ &= o\left(\sqrt{\frac{\log n}{2n}}\right).\end{aligned}$$

The last estimate proves (16). ■

4 Restricted 2-covers and line graphs: an analytic approach

Our proof of (2) will use generating function analysis. Let $a_{n,m}$ be the number of restricted, proper 2-covers on $[n]$ with m blocks. The generating function

for restricted, proper 2-covers

$$A(x, y) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n} \frac{a_{n,m}}{n!} x^n y^m$$

equals

$$A(x, y) = \exp\left(-y - \frac{xy^2}{2}\right) \sum_{m \geq 0} \frac{y^m}{m!} (1+x)^{\binom{m}{2}}; \quad (17)$$

see page 203 of [4]. A brief proof of (17) is that $(1+x)^{\binom{m}{2}}$ is the generating function for labelled graphs on m vertices and so $\sum_{m \geq 0} \frac{y^m}{m!} (1+x)^{\binom{m}{2}}$ is the exponential generating function of labelled graphs. Now, the factor $\exp(-y - xy^2/2)$ forbids isolated vertices and isolated edges.

Therefore,

$$V(x) = A(x, 1) = e^{-1} \sum_{m=0}^{\infty} \frac{1}{m!} (1+x)^{\binom{m}{2}} e^{-x/2} \quad (18)$$

and

$$\begin{aligned} v_n &= n! e^{-1} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^n \frac{1}{k!} \left(-\frac{1}{2}\right)^k \binom{\binom{m}{2}}{n-k} \\ &= n! e^{-1} \sum_{m=2}^{\infty} \frac{m^{2n}}{m!} \sum_{k=0}^n \frac{1}{k!} \left(-\frac{1}{2}\right)^k m^{-2n} \binom{\binom{m}{2}}{n-k} + o(1). \end{aligned} \quad (19)$$

Note that for $m \geq 2$,

$$\begin{aligned} \left| \sum_{k=0}^n \frac{n!}{k!} \left(-\frac{1}{2}\right)^k m^{-2n} \binom{\binom{m}{2}}{n-k} \right| &\leq \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{2}\right)^k m^{-2n} \binom{m}{2}^{n-k} \\ &\leq 2^{-n} \sum_{k=0}^n \binom{n}{k} m^{-2k} \\ &= 2^{-n} \left(\frac{1+m^{-2}}{2}\right)^n = O(2^{-n}). \end{aligned} \quad (20)$$

We will make use of the asymptotic analysis of the Bell numbers in Example 5.4 of [8], which uses the identity

$$B_n = e^{-1} \sum_{m=0}^{\infty} \frac{m^n}{m!}.$$

Let m_0 be the nearest integer to $\frac{2n}{W(2n)}$, where W is defined by (3). (The choice of m_0 is slightly different here than in [8], but the analysis giving (21) and (22) below remains valid.) In [8] it is proved that

$$\sum_{\substack{1 \leq m \leq n \\ |m-m_0| > \sqrt{n} \log n}} \frac{m^{2n}}{m!} = O\left(\frac{m_0^{2n}}{m_0!} \sqrt{n} \exp(-(\log n)^3)\right) \quad (21)$$

and that

$$\begin{aligned} \sum_{\substack{1 \leq m \leq n \\ |m-m_0| \leq \sqrt{n} \log n}} \frac{m^{2n}}{m!} &= \frac{m_0^{2n+1}}{m_0!} \sqrt{\frac{2\pi}{2n+m_0}} (1 + O((\log n)^6 n^{-1/2})) \quad (22) \\ &\sim eB_{2n}. \quad (23) \end{aligned}$$

It follows from (20) and (21) that

$$\begin{aligned} &\sum_{\substack{1 \leq m \leq n \\ |m-m_0| > \sqrt{n} \log n}} \frac{m^{2n}}{m!} \sum_{k=0}^n \frac{n!}{k!} \left(-\frac{1}{2}\right)^k m^{-2n} \binom{\binom{m}{2}}{n-k} \\ &= O\left(\frac{m_0^{2n}}{m_0!} \sqrt{n} 2^{-n} \exp(-(\log n)^3)\right) \\ &= O\left(B_{2n} 2^{-n} \exp\left(-\frac{(\log n)^3}{2}\right)\right). \quad (24) \end{aligned}$$

We have

$$\begin{aligned} &\sum_{\substack{1 \leq m \leq n \\ |m-m_0| \leq \sqrt{n} \log n}} \frac{m^{2n}}{m!} \sum_{k=0}^n \frac{n!}{k!} \left(-\frac{1}{2}\right)^k m^{-2n} \binom{\binom{m}{2}}{n-k} \\ &= \sum_{\substack{1 \leq m \leq n \\ |m-m_0| \leq \sqrt{n} \log n}} \frac{m^{2n}}{m!} m^{-2n} n! \binom{\binom{m}{2}}{n} + \Delta, \quad (25) \end{aligned}$$

where

$$\Delta := \sum_{\substack{1 \leq m \leq n \\ |m-m_0| \leq \sqrt{n} \log n}} \frac{m^{2n}}{m!} \sum_{k=1}^n \frac{n!}{k!} \left(-\frac{1}{2}\right)^k m^{-2n} \binom{\binom{m}{2}}{n-k}$$

is bounded by

$$\begin{aligned}
|\Delta| &\leq \sum_{\substack{1 \leq m \leq n \\ |m-m_0| \leq \sqrt{n} \log n}} \frac{m^{2n}}{m!} \sum_{k=1}^n \frac{n!}{k!} m^{-2n} \binom{\binom{m}{2}}{n} \left(\frac{n}{\binom{m}{2} - n} \right)^k \\
&= O\left(\frac{\log^2 n}{n}\right) \sum_{\substack{1 \leq m \leq n \\ |m-m_0| \leq \sqrt{n} \log n}} \frac{m^{2n}}{m!} m^{-2n} n! \binom{\binom{m}{2}}{n}.
\end{aligned}$$

One may show that uniformly for m in the range $|m - m_0| \leq \sqrt{n} \log n$

$$m^{-2n} \binom{\binom{m}{2}}{n} n! = 2^{-n} \exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right) (1 + O(n^{-1/2} \log^6 n)),$$

hence,

$$|\Delta| = O\left(\frac{\log^2 n}{n}\right) 2^{-n} \exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right) B_{2n}. \quad (26)$$

The main term of (25) is

$$\begin{aligned}
&\sum_{\substack{1 \leq m \leq n \\ |m-m_0| \leq \sqrt{n} \log n}} \frac{m^{2n}}{m!} m^{-2n} n! \binom{\binom{m}{2}}{n} \\
&= 2^{-n} \exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right) (1 + o(1)) \sum_{\substack{1 \leq m \leq n \\ |m-m_0| \leq \sqrt{n} \log n}} \frac{m^{2n}}{m!} \\
&= e B_{2n} 2^{-n} \exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right) (1 + o(1)) \\
&= e B_{2n} \frac{1}{2^n \sqrt{n}} e^{-\left(\frac{1}{2} \log(2n/\log n)\right)^2} (1 + o(1))
\end{aligned} \quad (27)$$

where we have used the asymptotic expansion (4) and the definition of m_0 at the last step. Now (19), (24), (26) and (27) prove (2) for v_n .

In the previous argument the result would have been the same if the $e^{-x/2}$ in (18) were replaced by 1 because in the Taylor expansion of $e^{-x/2}$ the constant term 1 corresponds to the main term of (25) and the higher order terms contribute to Δ , which is negligible. The arguments for restricted

partitions and line graphs are similar, starting from the identities obtained from Proposition 5 and (18)

$$U(x) = e^{-1} \sum_{m=0}^{\infty} \frac{1}{m!} (1+x)^{\binom{m}{2}} e^{x/2}.$$

and

$$L(x) = e^{-1} \sum_{m=0}^{\infty} \frac{1}{m!} (1+x)^{\binom{m}{2}} e^{x/2 - x^3/6 - x^4/4 - x^5/8 - x^6/48}.$$

In each case only the contribution of the constant term of the Taylor expansion of the exponential is 1 and the remaining terms contribute to a quantity like Δ which is asymptotically insignificant. ■

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