Preface

It is common now in academic circles to lament the decline in the teaching of geometry in our schools and universities, and the resulting loss of "geometric intuition" among our students. On the other hand, recent decades have seen renewed links between geometry and physics, to the benefit of both disciplines. One of the world's leading mathematicians has argued that the insights of "pre-calculus" geometry have a rôle to play at all levels of mathematical activity (Arnol'd [A]). There is no doubt that a combination of the axiomatic and the descriptive approaches associated with algebra and geometry respectively can help avoid the worst excesses of either approach alone.

These notes are about geometry, but by no means all or even most of geometry. I am concerned with the geometry of incidence of points and lines, over an arbitrary field, and unencumbered by metrics or continuity (or even betweenness). The major themes are the projective and affine spaces, and the polar spaces associated with sesquilinear or quadratic forms on projective spaces. The treatment of these themes blends the descriptive (*What do these spaces look like?*) with the axiomatic (*How do I recognize them?*) My intention is to explain and describe, rather than to give detailed argument for every claim. Some of the theorems (especially the characterisation theorems) are long and intricate. In such cases, I give a proof in a special case (often over the field with two elements), and an outline of the general argument.

The classical works on the subject are the books of Dieudonné [L] and Artin [B]. I do not intend to compete with these books. But much has happened since they were written (the axiomatisation of polar spaces by Veldkamp and Tits (see Tits [S]), the classification of the finite simple groups with its many geometric spin-offs, Buekenhout's geometries associated with diagrams, etc.), and I have included some material not found in the classical books.

Roughly speaking, the first five chapters are on projective spaces, the last five on polar spaces. In more detail: Chapter 1 introduces projective and affine spaces synthetically, and derives some of their properties. Chapter 2, on projective planes, discusses the rôle of Desargues' and Pappus' theorems in the coordinatisation of planes, and gives examples of non-Desarguesian planes. In Chapter 3, we turn to the coordinatisation of higher-dimensional projective spaces, following Veblen and Young. Chapter 4 contains miscellaneous topics: recognition of some subsets of projective spaces, including conics over finite fields of odd characteristic (Segre's theorem); the structure of projective lines; and generation and simplicity of the projective special linear groups. Chapter 5 outlines Buekenhout's approach to geometry via diagrams, and illustrates by interpreting the earlier characterisation theorems in terms of diagrams.

Chapter 6 relates polarities of projective spaces to reflexive sesquilinear forms, and gives the classification of these forms. Chapter 7 defines polar spaces, the geometries associated with such forms, and gives a number of these properties; the Veldkamp–Tits axiomatisation and the variant due to Buekenhout and Shult are also discussed, and proved for hyperbolic quadrics and for quadrics over the 2-element field. Chapter 8 discusses two important low-dimensional phenomena, the Klein quadric and triality, proceeding as far as to define the polarity defining the Suzuki–Tits ovoids and the generalised hexagon of type G_2. In Chapter 9, we take a detour to look at the geometry of the Mathieu groups. This illustrates that there are geometric objects satisfying axioms very similar to those for projective and affine spaces, and also having a high degree of symmetry. In the final chapter, we define spinors and use them to investigate the geometry of dual polar spaces, especially those of hyperbolic quadrics.

The notes are based on postgraduate lectures given at Queen Mary and Westfield College in 1988 and 1991. I am grateful to members of the audience on these occasions for their comments and especially for their questions, which forced me to think things through more carefully than I might have done. Among many pleasures of preparing these notes, I count two lectures by Jonathan Hall on his beautiful proof of the characterisation of quadrics over the 2-element field, and the challenge of producing the diagrams given the constraints of the typesetting system!

In the introductory chapters to both types of spaces (Chapters 1 and 6), as well as elsewhere in the text (especially Chapter 10), some linear algebra is assumed. Often, it is necessary to do linear algebra over a non-commutative field; but the differences from the commutative case are discussed. A good algebra textbook (for example, Cohn (1974)) will contain what is necessary.

Peter J. Cameron, London, 1991

Preface to the second edition

Materially, this edition is not very different from the first edition which was published in the QMW Maths Notes series in 1991. I have converted the files into LaTeX, corrected some errors, and added some new material and a few more references; this version does not represent a complete bringing up-to-date of the original. I intend to publish these notes on the Web.

In the meantime, one important relevant reference has appeared: Don Taylor's book *The Geometry of the Classical Groups* [R]. (Unfortunately, it has already gone out of print!) You can also look at my own lecture notes on Classical Groups (which can be read in conjunction with these notes, and which might be integrated with them one day). Other sources of information include the *Handbook of Incidence Geometry* [E] and (on the Web) two series of SOCRATES lecture notes at
`http://dwispc8.vub.ac.be/Potenza/lectnotes.html`
and
`http://cage.rug.ac.be/~fdc/intensivecourse2/final.html`

Please note that, in Figure 2.3, there are a few lines missing: dotted lines utq and urv and a solid line ub_1c_2. (The reason for this is hinted at in Exercise 3 in Section 1.2.)

Peter J. Cameron, London, 2000

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