8

The Klein quadric and triality

Low-dimensional hyperbolic quadrics possess a remarkably rich structure; the Klein quadric in 5-space encodes a projective 3-space, and the triality quadric in 7-space possesses an unexpected threefold symmetry. The contents of this chapter can be predicted from the diagrams of these geometries, since D_3 is isomorphic to A_3, and D_4 has an automorphism of order 3.

8.1 The Pfaffian

The determinant of a skew-symmetric matrix is a square. This can be seen in small cases by direct calculation:

$$\det\begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix} = a_{12}^2,$$

$$\det\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2.$$

Theorem 8.1 *(a) The determinant of a skew-symmetric matrix of odd size is zero.*

(b) There is a unique polynomial $\mathrm{Pf}(A)$ *in the indeterminates* a_{ij} *for* $1 \leq i < j \leq 2n$*, having the properties*

(i) if A *is a skew-symmetric* $2n \times 2n$ *matrix with* (i, j) *entry* a_{ij} *for* $1 \leq i < j \leq 2n$*, then*

$$\det(A) = \mathrm{Pf}(A)^2;$$

(ii) $\mathrm{Pf}(A)$ *contains the term* $a_{12}a_{34}\cdots a_{2n-1\,2n}$ *with coefficient* $+1$.

Proof We begin by observing that, if A is a skew-symmetric matrix, then the form B defined by

$$B(x,y) = xAy^\top$$

is an alternating bilinear form. Moreover, B is non-degenerate if and only if A is non-singular: for $xAy^\top = 0$ for all y if and only if $xA = 0$. We know that there is no non-degenerate alternating bilinear form on a space of odd dimension; so (a) is proved.

We know also that, if A is singular, then $\det(A) = 0$, whereas if A is non-singular, then there exists an invertible matrix P such that

$$PAP^\top = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right),$$

so that $\det(A) = \det(P)^{-2}$. Thus, $\det(A)$ is a square in either case.

Now regard a_{ij} as being indeterminates over the field F; that is, let $K = F(a_{ij} : 1 \le i < j \le 2n)$ be the field of fractions of the polynomial ring in $n(2n-1)$ variables over F. If A is the skew-symmetric matrix with entries a_{ij} for $1 \le i < j \le 2n$, then as we have seen, $\det(A)$ is a square in K. It is actually the square of a polynomial. (For the polynomial ring is a unique factorisation domain; if $\det(A) = (f/g)^2$, where f and g are polynomials with no common factor, then $\det(A)g^2 = f^2$, and so f^2 divides $\det(A)$; this implies that g is a unit.) Now $\det(A)$ contains a term

$$a_{12}^2 a_{34}^2 \cdots a_{2n-1\,2n}^2$$

corresponding to the permutation

$$(1\,2)(3\,4)\cdots(2n-1\,2n),$$

and so by choice of sign in the square root we may assume that (ii)(b) holds. Clearly the polynomial $\mathrm{Pf}(A)$ is uniquely determined.

The result for arbitrary skew-symmetric matrices is now obtained by specialisation (that is, substituting values from F for the indeterminates a_{ij}). ■

Exercises

1. A *one-factor* on the set $\{1,2,\ldots,2n\}$ is a partition F of this set into n subsets of size 2. We represent each 2-set$\{i,j\}$ by the ordered pair (i,j) with $i<j$. The *crossing number* $\chi(F)$ of the one-factor F is the number of pairs $\{(i,j),(k,l)\}$ of sets in F for which $i<k<j<l$.

(a) Let $\mathcal{F}_n$ be the set of one-factors on the set $\{1,2,\ldots,2n\}$. What is $|\mathcal{F}_n|$?

(b) Let $A = (a_{ij})$ be a skew-symmetric matrix of order $2n$. Prove that

$$\mathrm{Pf}(A) = \sum_{F\in\mathcal{F}_n} (-1)^{\chi(F)} \prod_{(i,j)\in F} a_{ij}.$$

2. Show that, if A is a skew-symmetric matrix and P any invertible matrix, then

$$\mathrm{Pf}(PAP^\top) = \det(P)\cdot\mathrm{Pf}(A).$$

Hint: We have $\det(PAP^\top) = \det(P)^2\det(A)$, and taking the square root shows that $\mathrm{Pf}(PAP^\top) = \pm\det(P)\,\mathrm{Pf}(A)$; it is enough to justify the positive sign. Show that it suffices to consider the 'standard' skew-symmetric matrix

$$A = \mathrm{diag}\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ldots,\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right),$$

In this case, show that the $(2n-1,2n)$ entry in $PAP^\top$ contains the term $p_{2n-1\,2n-1}p_{2n\,2n}$, so that $\mathrm{Pf}(PAP^\top)$ contains the diagonal entry of $\det(P)$ with sign $+1$.

3. Show that any linear transformation of a vector space fixing a symplectic form (a non-degenerate alternating bilinear form) has determinant 1.

8.2 The Klein correspondence

We begin by describing an abstract polar space which appears not to be of classical type. Let F be a skew field, and consider the geometry $\mathcal{G}$ defined from $\mathrm{PG}(3,F)$ as follows:

- the POINTs of $\mathcal{G}$ are the lines of $\mathrm{PG}(3,F)$;
- the LINEs of $\mathcal{G}$ are the plane pencils (incident point-plane pairs);
- the PLANEs of $\mathcal{G}$ are of two types: the points, and the planes.

A POINT and LINE are incident if the line belongs to the plane pencil (i.e., is incident with both the point and the plane). A LINE and PLANE are incident if the point or plane is one of the elements of the incident pair; and incidence between a POINT and a PLANE is the usual incidence in $\mathrm{PG}(3,F)$.

If a PLANE is a plane Π, then the POINTs and LINEs of this PLANE correspond to the lines and points of Π; so the residue of the plane is isomorphic to the dual of Π, namely, $\mathrm{PG}(2,F^\circ)$. On the other hand, if a PLANE is a point p, then the POINTs and LINEs of this PLANE are the lines and planes through p, so its residue is the residue of p in $\mathrm{PG}(3,F)$, namely $\mathrm{PG}(2,F)$. Thus (PS1) holds. (Note that, if F is not isomorphic to its opposite, then the space contains non-isomorphic planes, something which cannot happen in a classical polar space.)

Axiom (PS2) is clear. Consider (PS3). Suppose that the PLANE in question is a plane Π, and the POINT not incident with it is a line L. Then $L\cap\Pi$ is a point p; the set of POINTs of Π collinear with L is the plane pencil defined by p and Π (which is a LINE), and the union of the LINEs joining them to L consists of all lines through p (which is a PLANE), as required. The other case is dual.

Finally, if the point p and plane Π are non-incident, then the PLANEs they define are disjoint, proving (PS4).

Note that any LINE is incident with just two PLANEs, one of each type; so, if the polar space is classical, it must be a hyperbolic quadric in $\mathrm{PG}(5,F)$. We now show that, if F is commutative, it is indeed this quadric in disguise! (For non-commutative fields, this is one of the exceptional rank 3 polar spaces mentioned in Section 7.6.)

The skew-symmetric matrices of order 4 over F form a vector space of rank 6, with $x_{12},\ldots,x_{34}$ as coordinates. The Pfaffian is a quadratic form on this vector space, which vanishes precisely on the singular matrices. So, projectively, the singular matrices form a quadric Q in $\mathrm{PG}(5,F)$, the so-called *Klein quadric*. From the form of the Pfaffian, we see that this quadric is hyperbolic — but in fact this will become clear geometrically.

Any skew-symmetric matrix has even rank. In our case, a non-zero singular skew-symmetric matrix A has rank 2, and so can be written in the form

$$A = X(\mathbf{v},\mathbf{w}) := \mathbf{v}^\top\mathbf{w} - \mathbf{w}^\top\mathbf{v}$$

for some vectors $\mathbf{v},\mathbf{w}$. Replacing these two vectors by linear combinations $\alpha\mathbf{v}+\beta\mathbf{w}$ and $\gamma\mathbf{v}+\delta\mathbf{w}$ multiplies A by a factor $\alpha\delta-\beta\gamma$ (which is just the determinant of the transformation). So we have a map from the line of $\mathrm{PG}(3,F)$ spanned by $\mathbf{v}$ and $\mathbf{w}$ to the point of the Klein quadric spanned by $X(\mathbf{v},\mathbf{w})$. This map is a bijection: we have seen that it is onto, and the matrix determines the line as its row space.

This bijection has the properties predicted by our abstract treatment. Most important,

> two points of the Klein quadric are perpendicular if and only if the corresponding lines intersect.

To prove this, note that two points are perpendicular if and only if the line joining them lies in Q. Now, if two lines intersect, we can take them to be $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\langle \mathbf{u}, \mathbf{w} \rangle$; and we have

$$\alpha(\mathbf{u}^\top \mathbf{v} - \mathbf{v}^\top \mathbf{u}) + \beta(\mathbf{u}^\top \mathbf{w} - \mathbf{w}^\top \mathbf{u}) = \mathbf{u}^\top (\alpha\mathbf{v} + \beta\mathbf{w}) - (\alpha\mathbf{v} + \beta\mathbf{w})^\top \mathbf{u},$$

so the line joining the corresponding points lies in the quadric. Conversely, if two lines are skew, then they are $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ and $\langle \mathbf{v}_3, \mathbf{v}_4 \rangle$, where $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is a basis; then the matrix

$$\mathbf{v}_1^\top \mathbf{v}_2 - \mathbf{v}_2^\top \mathbf{v}_1 + \mathbf{v}_3^\top \mathbf{v}_4 - \mathbf{v}_4^\top \mathbf{v}_3$$

has rank 4, and is a point on the line not on Q.

Hence the planes on the quadric correspond to maximal families of pairwise intersecting lines, of which there are two types: all lines through a fixed point; and all lines in a fixed plane. Moreover, the argument in the preceding paragraph shows that lines on Q do indeed correspond to plane pencils of lines in $\mathrm{PG}(3, F)$. This completes the identification.

Exercise

1. This exercise gives the promised identification of $\mathrm{PSL}(4,2)$ with the alternating group A_8.

Let V be the vector space of rank 6 over $\mathrm{GF}(2)$ consisting of the binary words of length 8 having even weight modulo the subspace Z consisting of the all-zero and all-1 words. Show that the function

$$f(\mathbf{v} + Z) = \tfrac{1}{2}\mathrm{wt}(\mathbf{v}) \pmod 2$$

is well-defined and is a quadratic form of rank 3 on V, whose zeros form the Klein quadric Q. Show that the symmetric group S_8 interchanges the two families of planes on Q, the subgroup fixing the two families being the alternating group A_8.

Use the Klein correspondence to show that A_8 is embedded as a subgroup of $\mathrm{PGL}(4,2) = \mathrm{PSL}(4,2)$. By calculating the orders of these groups, show that equality holds.

Remark The isomorphism between $\mathrm{PSL}(4,2)$ and A_8 can be used to give a solution to Kirkman's *schoolgirl problem*. This problem asks for a schedule for fifteen schoolgirls to walk in five groups of three every day for seven days, subject to the requirement that any two girls walk together in a group exactly once during the week.

The 7×5 groups of girls are thus the blocks of a 2-$(15,3,1)$ design. We will take this design to consist of the points and lines of $\mathrm{PG}(3,2)$. The problem is then to find a 'parallelism' or 'resolution', a partition of the lines into seven 'parallel classes' each consisting of five pairwise disjoint lines.

One way to find a parallel class is to consider the underlying vector space $V(4,2)$ as a vector space of rank 2 over $\mathrm{GF}(4)$. The five 'points' or rank 1 subspaces over $\mathrm{GF}(4)$ become five pairwise disjoint lines when we restrict the scalars to $\mathrm{GF}(2)$. Scalar multiplication by a primitive element of $\mathrm{GF}(4)$ is an automorphism of order 3, fixing all five lines, and commuting with a subgroup $\mathrm{SL}(2,4) \cong A_5$. Moreover, if two such automorphisms oof order 3 have a common fixed line, then they generate a $\{2,3\}$-group, since the stabiliser of a line in $\mathrm{GL}(4,2)$ is a $\{2,3\}$-group.

Now, in A_8, an element of order 3 commuting with a subgroup isomorphic to A_5 is necessarily a 3-cycle. Two 3-cycles generate a $\{2,3\}$-group if and only if their supports intersect in 0 or 2 points. So we require a set of seven 3-subsets of $\{1,\ldots,8\}$, any two of which meet in one point. The lines of $\mathrm{PG}(2,2)$ (omitting one point) have this property.

8.3 Some dualities

We have interpreted points of the Klein quadric in $\mathrm{PG}(3,F)$. What about the points off the quadric?

Theorem 8.2 *There is a bijection from the set of points p outside the Klein quadric $\mathcal{Q}$ to symplectic structures on $\mathrm{PG}(3,F)$, with the property that a point of $\mathcal{Q}$ perpendicular to p translates under the Klein correspondence to a totally isotropic line for the symplectic geometry.*

Proof A point $p \notin \mathcal{Q}$ is represented by a skew-symmetric matrix A which has non-zero Pfaffian (and hence is invertible), up to a scalar multiple. The matrix defines a symplectic form b, by the rule

$$b(\mathbf{v},\mathbf{w}) = \mathbf{v}A\mathbf{w}^\top.$$

We must show that a line is t.i. with respect to this form if and only if the corresponding point of $\mathcal{Q}$ is perpendicular to p.

Let A be a non-singular skew-symmetric 4×4 matrix over a field F. By direct calculation, we show that the following assertions are equivalent, for any vectors $\mathbf{v},\mathbf{w}\in F^4$:

(a) $X(\mathbf{v},\mathbf{w})=\mathbf{v}^\top\mathbf{w}-\mathbf{w}^\top\mathbf{v}$ is orthogonal to A, with respect to the bilinear form obtained by polarising the quadratic form $Q(X)=\mathrm{Pf}(X)$;

(b) $\mathbf{v}$ and $\mathbf{w}$ are orthogonal with respect to the symplectic form with matrix $A^\dagger$, that is, $\mathbf{v}A^\dagger\mathbf{w}^\top=0$.

Here the matrices A and $A^\dagger$ are given by

$$A=\begin{pmatrix}0 & a_{12} & a_{13} & a_{14}\\ -a_{12} & 0 & a_{23} & a_{24}\\ -a_{13} & -a_{23} & 0 & a_{34}\\ -a_{14} & -a_{24} & -a_{34} & 0\end{pmatrix},\qquad A^\dagger=\begin{pmatrix}0 & a_{34} & -a_{24} & a_{23}\\ -a_{34} & 0 & a_{14} & -a_{13}\\ a_{24} & -a_{14} & 0 & a_{12}\\ -a_{23} & a_{13} & -a_{12} & 0\end{pmatrix}.$$

Note that, if A is the matrix of the standard symplectic form, then so is $A^\dagger$. In general, the map taking the point outside the quadric spanned by A to the symplectic form with matrix $A^\dagger$ is the one asserted in the theorem. ■

Let $\mathcal{G}_1$ be the symplectic GQ over F, and $\mathcal{G}_2$ the orthogonal GQ associated with the quadric $\mathbf{v}^\perp\cap\mathcal{Q}$, where $\mathcal{Q}$ is the Klein quadric and $\langle\mathbf{v}\rangle\notin\mathcal{Q}$. (Note that any non-singular quadratic form of rank 2 in 5 variables is equivalent to $\alpha x_0^2+x_1x_2+x_3x_4$ for some $\alpha\neq 0$; so any two such forms are equivalent up to scalar multiple, and define the same GQ.) We have defined a map from points of $\mathcal{G}_2$ to lines of $\mathcal{G}_1$. Given any point p of $\mathcal{G}_1$, the lines of $\mathcal{G}_1$ containing p form a plane pencil in $\mathrm{PG}(3,F)$, and so translate into a line of $\mathcal{G}_2$. Thus we have shown:

Theorem 8.3 *For any field F, the symplectic GQ in $\mathrm{PG}(3,F)$ and the orthogonal GQ in $\mathrm{PG}(4,F)$ are dual.* ■

Now let F be a field which has a Galois extension K of degree 2 and σ the Galois automorphism of K over F. With the extension K/F we can associate two GQs:

$\mathcal{G}_1'$: the unitary GQ in $\mathrm{PG}(3,K)$, defined by the Hermitian form

$$x_1y_2^\sigma+x_2y_1^\sigma+x_3y_4^\sigma+x_4y_3^\sigma;$$

$\mathcal{G}_2'$: the orthogonal GQ in $\mathrm{PG}(5,F)$ defined by the quadratic form

$$x_1x_2+x_3x_4+\alpha x_5^2+\beta x_5x_6+\gamma x_6^2,$$

where $\alpha x^2+\beta x+\gamma$ is an irreducible quadratic over F which splits in K.

Theorem 8.4 *The two GQs $\mathcal{G}_1'$ and $\mathcal{G}_2'$ defined above are dual.*

Proof This is proved by "twisting the Klein correspondence". In outline, we take the Klein correspondence over K, and change coordinates on the quadric so that restriction of scalars to F gives the geometry $\mathcal{G}_2'$, rather than the Klein quadric over F; then show that the corresponding set of lines in $\mathrm{PG}(3,K)$ are those which are totally isotropic with respect to a Hermitian form. ∎

Exercises

1. Prove the assertion about A and $A^\dagger$ in the proof of Theorem 8.2.

Let $\mathcal{Q}$ be a hyperbolic quadric of rank n. If v is a non-singular vector, then the quadric $v^\perp \cap \mathcal{Q} = \mathcal{S}$ has the property

- $\mathcal{S}$ meets every maximal subspace E of $\mathcal{Q}$ in a hyperplane of E.

We call a set $\mathcal{S}$ satisfying this condition *special*. The point of the next three exercises is to investigate whether special sets are necessarily quadrics of the form $v^\perp \cap \mathcal{Q}$.

2. Consider the case $n = 2$. Let the rank 4 vector space be the space of all 2×2 matrices over F, and let the quadratic form be the determinant.

(a) Show that the map
$$\langle X \rangle \mapsto (\mathrm{Ker}(X), \mathrm{Im}(X))$$
induces a bijection between the point set of the quadric $\mathcal{Q}$ and $P \times P$, where P is the projective line over F.

(b) If A is a non-singular matrix, show that
$$A^\perp = \{\langle X \rangle \in \mathcal{Q} : \mathrm{Ker}(X) \cdot A = \mathrm{Im}(X)\},$$
which corresponds under this bijection to the set $\{(p, p \cdot A) : p \in P\}$.

(c) Show that, if π is any permutation of P, then $\{(p, \pi(p)) : p \in P\}$ is a special set; and all special sets have this form.

(d) Deduce that every special set is a quadric if and only if $|F| \le 3$.

3. Consider the case $n = 3$. Take $\mathcal{Q}$ to be the Klein quadric. Show that the Klein correspondence maps the special set $\mathcal{S}$ to a set S of lines of $\mathrm{PG}(3,F)$ with the property that the set of lines of S through any point of p, or the set of lines of S in any plane Π, is a plane pencil. Show that the correspondence $p \mapsto \Pi$ of $\mathrm{PG}(3,F)$, where the set of lines of S containing p and the set contained in Π are equal, is a symlectic polarity with S as its set of absolute lines. Deduce that S is the set of lines of a symplectic GQ in $\mathrm{PG}(3,F)$, and hence that $\mathcal{S}$ is a quadric.

4. Prove by induction on n that, for $n \geq 3$, any special set is a quadric. (See Cameron and Kantor [12] for a crib.)

8.4 Dualities of symplectic quadrangles

A field of characteristic 2 is said to be *perfect* if every element is a square. A finite field of characteristic 2 is perfect, since the multiplicative group has odd order.

If F has characteristic 2, then the map $x \mapsto x^2$ is a homomorphism, since

$$\begin{aligned} (x+y)^2 &= x^2+y^2, \\ (xy)^2 &= x^2y^2, \end{aligned}$$

and is one-to-one. Hence F is perfect if and only if this map is an automorphism.

Theorem 8.5 *Let F be a perfect field of characteristic 2. Then there is an isomorphism between the symplectic polar space of rank n over F, and the orthogonal polar space of rank n defined by a quadratic form in $2n+1$ variables.*

Proof Let V be a vector space of rank $2n+1$ carrying a non-singular quadratic form f of rank n. By polarising f, we get an alternating bilinear form b, which cannot be non-degenerate; its radical $R = V^{\perp}$ is of rank 1, and the restriction of f to it is the germ of f.

Let W_0 be a totally singular subspace of V. Then $W = W_0 + R$ is a totally isotropic subspace of the non-degenerate symplectic space V/R. So we have an incidence-preserving injection $\theta : W_0 \mapsto (W_0+R)/R$ from the orthogonal polar space to the symplectic. We have to show that θ is onto.

So let W/R be t.i. This means that W itself is t.i. for the form b; but $R \subseteq W$, so W is not t.s. for f. However, on W, we have

$$\begin{aligned} f(\mathbf{w}_1+\mathbf{w}_2) &= f(\mathbf{w}_1)+f(\mathbf{w}_2), \\ f(\alpha\mathbf{w}) &= \alpha^2 f(\mathbf{w}), \end{aligned}$$

so f is semilinear on W. Thus, the kernel of f is a hyperplane W_0 of W. The space W_0 is t.s., and $W_0 + R = W$; so W_0 maps onto W/R under θ. ∎

Now consider the case $n = 2$. We have an isomorphism between the symplectic and orthogonal quadrangles, by Theorem 8.5, and a duality, by Theorem 8.3. So:

Theorem 8.6 *The symplectic generalised quadrangle over a perfect field of characteristic* 2 *is self-dual.* ∎

When is there a polarity?

Theorem 8.7 *Let F be a perfect field of characteristic* 2*. Then the symplectic GQ over F has a polarity if and only if F has an automorphism σ satisfying*

$$\sigma^2 = 2,$$

where 2 *denotes the automorphism $x \mapsto x^2$.*

Proof For this, we cannot avoid using coordinates! We take the vector space F^4 with the standard symplectic form

$$b((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)) = x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3.$$

(Remember that the characteristic is 2.) The Klein correspondence takes the line spanned by (x_1,x_2,x_3,x_4) and (y_1,y_2,y_3,y_4) to the point with coordinates z_{ij}, $1 \le i < j \le 4$, where $z_{ij} = x_iy_j + x_jy_i$; this point lies on the quadric with equation

$$z_{12}z_{34} + z_{13}z_{24} + z_{14}z_{23} = 0,$$

and (if the line is t.i.) also on the hyperplane $z_{12} + z_{34} = 0$. If we factor out the subspace spanned by the point with $z_{12} = z_{34} = 1$, $z_{ij} = 0$ otherwise, and use coordinates $(z_{13},z_{24},z_{14},z_{23})$, we obtain a point of the symplectic space; the map δ from lines to points is the duality previously defined.

To compute the image of a point $p = (a_1,a_2,a_3,a_4)$ under the duality, take two t.i. lines through this point and calculate their images. If a_1 and a_4 are non-zero, we can use the lines joining p to the points $(a_1,a_2,0,0)$ and $(0,a_4,a_1,0)$; the images are $(a_1a_3,a_2a_4,a_1a_4,a_2a_3)$ and $(a_1^2,a_4^2,0,a_1a_2+a_3a_4)$. Now the image of the line joining these points is found to be the point $(a_1^2,a_2^2,a_3^2,a_4^2)$. The same formula is found in all cases. So δ^2 is the collineation induced by the field automorphism $x \mapsto x^2$, or 2 as we have called it.

Suppose that there is a field automorphism σ with $\sigma^2 = 2$, and let $\theta = \sigma^{-1}$; then $(\delta\theta)^2$ is the identity, so $\delta\theta$ is a polarity.

Conversely, suppose that there is a polarity. By Theorem 7.14, any collineation g is induced by the product of a linear transformation and a uniquely defined field automorphism $\theta(g)$. Now any duality has the form δg for some collineation g; and

$$\theta((\delta g)^2) = 2\theta(g)^2.$$

So, if δg is a polarity, then $2\theta(g)^2 = 1$, whence $\sigma = \theta(g)^{-1}$ satisfies $\sigma^2 = 2$. ■

In the case where F is a finite field $\mathrm{GF}(2^m)$, the automorphism group of F is cyclic of order m, generated by 2; and so there is a solution of $\sigma^2 = 2$ if and only if m is odd. We conclude that the symplectic quadrangle over $\mathrm{GF}(2^m)$ has a polarity if and only if m is odd.

We now examine the set of *absolute* points and lines (i.e., those incident with their image). A *spread* is a set S of lines such that every point lies on a unique line of S. Dually, an *ovoid* in a GQ is a set O of points with the property that any line contains a unique point of O. Note that this is quite different from the definition of an ovoid in $\mathrm{PG}(3,F)$ given in Section 4.4; but there is a connection, as we will see.

Proposition 8.8 *The set of absolute points of a polarity of a GQ is an ovoid, and the set of absolute lines is a spread.*

Proof Let δ be a polarity. No two absolute points are collinear. For, if x and y are absolute points lying on the line L, then x, y and $L\delta$ would form a triangle.

Suppose that the line L contains no absolute point. Then L is not absolute, so $L\delta \notin L$. Thus, there is a unique line M containing $L\delta$ and meeting L. Then $M\delta \in L$, so $M\delta$ is not absolute. But L meets M, so $L\delta$ and $M\delta$ are collinear; hence $L\delta, M\delta$ and $L \cap M$ form a triangle.

The second statement is dual. ■

Theorem 8.9 *The set of absolute points of a polarity of a symplectic GQ in* $\mathrm{PG}(3,F)$ *is an ovoid in* $\mathrm{PG}(3,F)$.

Proof Let σ be the polarity of the GQ $\mathcal{G}$, and $\perp$ the polarity of the projective space defining the GQ. By the last result, the set O of absolute points of σ is an ovoid in $\mathcal{G}$. This means that the t.i. lines are tangents to O, and the t.i. lines through a point of O form a plane pencil. So we have to prove that any other line of the projective space meets O in 0 or 2 points.

Let X be a hyperbolic line, p a point of $X \cap O$, and $p^\sigma = L$. Then L meets the hyperbolic line $L^\perp$ in a point q. Let $q^\sigma = M$. Since $q \in L$, we have $p \in M$; so M also meets $X^\perp$, in a point r. Let $N = r^\sigma$. Then $q \in N$, so N meets X. Also, N meets O in a point s. The line s^σ contains s and $N^\sigma = r$. So s is on two lines meeting $X^\perp$, whence $s \in X$. So, if $|X \cap O| \geq 1$, then $|X \cap O| \geq 2$.

Now let p' be another point of $X \cap O$, and define L' and q' as before. Let K be the line pq'. Then $p \in K$, so $p^\sigma = L$ contains $x = K^\sigma$. Also, K meets L', so x is collinear with p'. But the only point of L collinear with p' is q. So $x = q$, independent of p'. This means that there is only one point $p' \neq p$ in $X \cap O$, and this set has cardinality 2. ∎

Remark Over finite fields, any ovoid in a symplectic GQ is an ovoid in the ambient projective 3-space. This is false for infinite fields. (See Exercises 2 and 3.)

Hence, if F is a perfect field of characteristic 2 in which $\sigma^2 = 2$ for some automorphism σ, then $\mathrm{PG}(3,F)$ possesses symplectic ovoids and spreads. These give rise to inversive planes and to translation planes, as described in Sections 4.1 and 4.4. For finite fields F, these are the only known ovoids other than elliptic quadrics.

Exercises

1. Suppose that the points and lines of a GQ are all the points and some of the lines of $\mathrm{PG}(3,F)$. Prove that the lines through any point form a plane pencil, and deduce that the GQ is symplectic.

2. Prove that an ovoid O in a symplectic GQ over the finite field $\mathrm{GF}(q)$ is an ovoid in $\mathrm{PG}(3,q)$. [Hint: as in Theorem 8.3.5, it suffices to prove that any hyperbolic line meets O in 0 or 2 points. Now, if X is a hyperbolic line with $X \cap O \neq \emptyset$, then $X^\perp \cap O = \emptyset$, so at most half of the $q^2(q^2+1)$ hyperbolic lines meet O. Take any $N = \frac{1}{2}q^2(q^2+1)$ hyperbolic lines including all those meeting O, and let n_i of the chosen lines meet O in i points. Prove that $\sum n_i = N$, $\sum i n_i = 2N$, $\sum i(i-1)n_i = 2N$.]

3. Prove that, for any infinite field F, there is an ovoid of the symplectic quadrangle over F which is not an ovoid of the embedding projective space.

8.5 Reguli and spreads

We met in Section 4.1 the concepts of a regulus in $\mathrm{PG}(3,F)$ (the set of common transversals to three pairwise skew lines), a spread (a set of pairwise skew lines covering all the points), a bispread (a spread containing a line of each plane), and a regular spread (a spread containing the regulus through any three of its lines). We now translate these concepts to the Klein quadric.

Theorem 8.10 *Under the Klein correspondence,*

(a) a regulus corresponds to a conic, the intersection if $\mathcal{Q}$ with a non-singular plane Π, and the opposite regulus to the intersection of $\mathcal{Q}$ with $\Pi^\perp$;

(b) a bispread corresponds to an ovoid, a set of pairwise non-perpendicular points meeting every plane on $\mathcal{Q}$;

(c) a regular spread corresponds to the ovoid $\mathcal{Q}\cap W^\perp$, where W is a line disjoint from $\mathcal{Q}$.

Proof (a) Take three pairwise skew lines. They translate into three pairwise non-perpendicular points of $\mathcal{Q}$, which span a non-singular plane Π (so that $\mathcal{Q}\cap\Pi$ is a conic C). Now $\Pi^\perp$ is also a non-singular plane, and $\mathcal{Q}\cap\Pi^\perp$ is a conic C', consisting of all points perpendicular to the three given points. Translating back, C' corresponds to the set of common transversals to the three given lines. This set is a regulus, and is opposite to the regulus spanned by the given lines (corresponding to C).

(b) This is straightforward translation. Note, incidentally, that a spread (or a cospread) corresponds to what might be called a “semi-ovoid”, were it not that this term is used for a different concept: that is, a set of pairwise non-perpendicular points meeting every plane in one family on $\mathcal{Q}$.

(c) A regular spread is “generated” by any four lines not contained in a regulus, in the sense that it is obtained by repeatedly adjoining all the lines in a regulus through three of its lines. On $\mathcal{Q}$, the four given lines translate into four points, and the operation of generation leaves us within the 3-space they span. This 3-space has the form $W^\perp$ for some line W; and no point of $\mathcal{Q}$ can be perpendicular to every point of such a 3-space. ■

Note that a line disjoint from $\mathcal{Q}$ is anisotropic; such lines exist if and only if there is an irreducible quadratic over F, that is, F is not quadratically closed. (We saw earlier the construction of regular spreads: if K is a quadratic extension of F, take the rank 1 subspaces of a rank 2 vector space over K, and restrict scalars to F.)

Thus a bispread is regular if and only if the corresponding ovoid is contained in a 3-space section of $\mathcal{Q}$. A bispread whose ovoid lies in a 4-space section of $\mathcal{Q}$ is called *symplectic*, since its lines are totally isotropic with respect to some symplectic form (by the results of Section 8.3). An open problem is to find a simple structural test for symplectic bispreads (resembling the characterisation of regular spreads in terms of reguli).

We also saw in Section 4.1 that spreads of lines in projective space give rise to translation planes; and regular spreads give Desarguesian (or Pappian) planes. Another open problem is to characterise the translation planes arising from symplectic spreads or bispreads.

8.6 Triality

Now we increase the rank by 1, and let $\mathcal{Q}$ be a hyperbolic quadric in $\mathrm{PG}(7,F)$, defined by a quadratic form of rank 4. The maximal t.s. subspaces have dimension 3, and are called *solids*; as usual, they fall into two families $\mathcal{M}_1$ and $\mathcal{M}_2$, so that two solids in the same family meet in a line or are disjoint, while two solids in different families meet in a plane or a point. Any t.s. plane lies in a unique solid of each type. Let $\mathcal{P}$ and $\mathcal{L}$ be the sets of points and lines.

Consider the geometry defined as follows.

- The POINTs are the elements of $\mathcal{M}_1$.
- The LINEs are the elements of $\mathcal{L}$.
- The PLANEs are incident pairs (p,M), $p \in \mathcal{P}$, $M \in \mathcal{M}_2$.
- The SOLIDs are the elements of $\mathcal{P} \cup \mathcal{M}_2$.

Incidence is defined as follows. Between POINTs, LINEs and SOLIDs, it is as in the quadric, with the additional rule that the POINT M_1 and SOLID M_2 are incident if they intersect in a plane. The PLANE (p,M) is incident with all those varieties incident with both p and M.

Proposition 8.11 *The geometry just described is an abstract polar space in which any PLANE is incident with just two SOLIDs.*

Proof We consider the axioms in turn.

(P1): Consider, for example, the SOLID $M \in \mathcal{M}_2$. The POINTs incident with M are bijective with the planes of M; the LINEs are the lines of M; the PLANEs are pairs (p, M) with $p \in M$, and so are bijective with the points of M. Incidence is defined so as to make the subspaces contained in M a projective space isomorphic to the dual of M.

For the SOLID $p \in \mathcal{P}$, the argument is a little more delicate. The geometry $p^\perp/p$ is a hyperbolic quadric in $\mathrm{PG}(5,F)$, that is, the Klein quadric; the POINTs, LINEs and PLANEs incident with p are bijective with one family of planes, the lines, and the other family of planes on the quadric; and hence (by the Klein correspondence) with the points, lines and planes of $\mathrm{PG}(3,F)$.

The other cases are easier.

(P2) is trivial, (P3) routine, and (P4) is proved by observing that if $p \in \mathcal{P}$ and $M \in \mathcal{M}_2$ are not incident, then no POINT can be incident with both.

Finally, the SOLIDs containing the PLANE (p, M) are p and M only. ∎

So the new geometry we constructed is itself a hyperbolic quadric in $\mathrm{PG}(7,F)$, and hence isomorphic to the original one. This implies the existence of a map τ which carries $\mathcal{L}$ to itself and $\mathcal{P} \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{P}$. This map is called a *triality* of the quadric, by analogy with dualities of projective spaces.

It is more difficult to describe trialities in coordinates. An algebraic approach must wait until Chapter 10.

Exercise

1. Prove the Buekenhout-Shult property for the geometry constructed in this section. That is, let $M \in \mathcal{M}_1$, $L \in \mathcal{L}$, and suppose that L is not incident with M; prove that either all members of $\mathcal{M}_1$ containing L meet M in a plane, or just one does, depending on whether L is disjoint from M or not.

8.7 An example

In this section we apply triality to the solution of a combinatorial problem first posed and settled by Breach and Street [2]. Our approach follows Cameron and Praeger [13].

Consider the set of planes of $\mathrm{AG}(3,2)$. They form a 3-$(8,4,1)$ design, that is, a collection of fourteen 4-subsets of an 8-set, any three points contained in exactly one of them. There are $\binom{8}{4} = 70$ 4-subsets altogether; can they be partitioned into five copies of $\mathrm{AG}(3,2)$? The answer is “no”, as has been known since the time of Cayley. (In fact, there cannot be more than two disjoint copies of $\mathrm{AG}(3,2)$ on an 8-set; a construction will be given in the next chapter.) Breach and Street asked: what if we take a 9-set? This has $\binom{9}{4} = 126$ 4-subsets, and can conceivably be partitioned into nine copies of $\mathrm{AG}(3,2)$, each omitting one point. They proved:

Theorem 8.12 *There are exactly two non-isomorphic ways to partition the* 4-*subsets of a* 9-*set into nine copies of* $\mathrm{AG}(3,2)$. *Both admit* 2-*transitive groups.*

Proof First we construct the two examples.

1. Regard the 9-set as the projective line over $\mathrm{GF}(8)$. If any point is designated as the point at infinity, the remaining points form an affine line over $\mathrm{GF}(8)$, and hence (by restricting scalars) an affine 3-space over $\mathrm{GF}(2)$. We take the fourteen planes of this affine 3-space as one of our designs, and perform the same construction for each point to obtain the desired partition. This partition is invariant under the group $\mathrm{P\Gamma L}(2,8)$, of order $9\cdot 8\cdot 7\cdot 3 = 1512$. The automorphism group is the stabiliser of the object in the symmetric group; so the number of partitions isomorphic to this one is the index of this group in S_9, which is $9!/1512 = 240$.

2. Alternatively, the nine points carry the structure of affine plane over $\mathrm{GF}(3)$. Identifying one point as the origin, the structure is a rank 2 vector space over $\mathrm{GF}(3)$. Put a symplectic form b on the vector space. Now there are six 4-sets which are symmetric differences of two lines through the origin, and eight 4-sets of the form $\{\mathbf{v}\} \cup \{\mathbf{w} : b(\mathbf{v},\mathbf{w}) = 1\}$ for non-zero $\mathbf{v}$. It is readily checked that these fourteen sets form a 3-design. Perform this construction with each point designated as the origin to obtain a partition. This one is invariant under the group $\mathrm{ASL}(2,3)$ generated by the translations and $\mathrm{Sp}(2,3) = \mathrm{SL}(2,3)$, of order $9\cdot 8\cdot 3 = 216$, and there are $9!/216 = 1680$ partitions isomorphic to this one.

Now we show that there are no others. We use the terminology of coding theory. Note that the fourteen words of weight 4 supporting planes of $\mathrm{AG}(3,2)$, together with the all-0 and all-1 words, form the extended Hamming code of length 8 (the code we met in Section 3.2, extended by an overall parity check); it is the only *doubly-even self-dual code* of length 8 (that is, the only code $C = C^\perp$ with all weights divisible by 4).

Let V be the vector space of all words of length 9 and even weight. The function $f(\mathbf{v}) = \frac{1}{2}\mathrm{wt}(\mathbf{v}) \pmod 2$ is a quadratic form on V, which polarises to the usual dot product. Thus maximal t.s. subspaces for f are just doubly even self-dual codes, and their existence shows that f has rank 4 and so is the split form defining the triality quadric. (The quadric Q consists of the words of weight 4 and 8.)

Suppose we have a partition of the 4-sets into nine affine spaces. An easy counting argument shows that every point is excluded by just one of the designs. So if we associate with each design the word of weight 8 whose support is its point set, we obtain a solid on the quadric, and indeed a *spread* or partition of the quadric into solids.

All these solids belong to the same family, since they are pairwise disjoint. So we can apply the triality map and obtain a set of nine points which are pairwise non-collinear, that is, an *ovoid*. Conversely, any ovoid gives a spread. In fact, an ovoid gives a spread of solids of each family, by applying triality and its inverse. So the total number of spreads is twice the number of ovoids.

The nine words of weight 8 form an ovoid. Any ovoid is equivalent to this one. (Consider the Gram matrix of inner products of the vectors of an ovoid; this must have zeros on the diagonal and ones elsewhere.) The stabiliser of this ovoid is the symmetric group S_9. So the number of ovoids is the index of S_9 in the orthogonal group, which turns out to be 960. Thus, the total number of spreads is $1920 = 240 + 1680$, and we have them all! ■

8.8 Generalised polygons

Projective and polar spaces are important members of a larger class of geometries called *buildings*. Much of the importance of these derives from the fact that they are the “natural” geometries for arbitrary groups of Lie type, just as projective spaces are for linear groups and polar spaces for classical groups. The groups of Lie type include, in particular, all the non-abelian finite simple groups except for the alternating groups and the twenty-six sporadic groups. I do not intend to discuss buildings here — for this, see the lecture notes of Tits [S] or the recent books by Brown [C] and Ronan [P] — but will consider the rank 2 buildings, or *generalised polygons* as they are commonly known. These include the 2-dimensional projective and polar spaces (that is, projective planes and generalised quadrangles).

Recall that a rank 2 geometry has two types of varieties, with a symmetric incidence relation; it can be thought of as a bipartite graph. We use graph-theoretic terminology in the following definition. A rank 2 geometry is a *generalised n-gon* (where $n \geq 2$) if

(GP1) it is connected with diameter n and girth $2n$;

(GP2) for any variety x, there is a variety y at distance n from x.

It is left to the reader to check that, for $n = 2,3,4$, this definition coincides with that of a digon, generalised projective plane or generalised quadrangle respectively.

Let $\mathcal{G}$ be a generalised n-gon. The *flag geometry* of $\mathcal{G}$ has as POINTs the varieties of $\mathcal{G}$ (of both types), and as LINEs the flags of $\mathcal{G}$, with the obvious incidence between POINTs and LINEs. It is easily checked to be a generalised $2n$-gon in which every line has two points; and any generalised $2n$-gon with two points per line is the flag geometry of a generalised n-gon. In future, we usually assume that our polygons are *thick*, that is, have at least three varieties of one type incident with each variety of the other type. It is also easy to show that a thick generalised polygon has *orders*, that is, the number of points per line and the number of lines per point are both constant; and, if n is odd, then these two constants are equal. [Hint: in general, if varieties x and y have distance n, then each variety incident with x has distance $n-2$ from a unique variety incident with y, and *vice versa*.]

We let $s+1$ and $t+1$ denote the numbers of points per line or lines per point, respectively, with the proviso that either or both may be infinite. (If both are finite, then the geometry is finite.) The geometry is thick if and only if $s,t > 1$. The major theorem about finite generalised polygons is the *Feit–Higman theorem* (Feit and Higman [17]:

Theorem 8.13 *A thick generalised n-gon can exist only for* $n = 2,3,4,6$ *or* 8. ∎

In the course of the proof, Feit and Higman derive additional information:

- if $n = 6$, then st is a square;
- if $n = 8$, then $2st$ is a square.

Subsequently, further numerical restrictions have been discovered; for example:

- if $n = 4$ or $n = 8$, then $t \leq s^2$ and $s \leq t^2$;
- if $n = 6$, then $t \leq s^3$ and $s \leq t^3$.

In contrast to the situation for $n = 3$ and $n = 4$, the only known finite thick generalised 6-gons and 8-gons arise from groups of Lie type. There are 6-gons with $s = t = q$ and with $s = q$, $t = q^3$ for any prime power q; and 8-gons with $s = q$, $t = q^2$, where q is an odd power of 2. In the next section, we discuss a class of 6-gons including the first-mentioned finite examples.

There is no hope of classifying infinite generalised n-gons, which exist for all n (Exercise 2). However, assuming a symmetry condition, the *Moufang condition*, which generalises the existence of central collineations in projective planes, and is also equivalent to a generalisation of Desargues' theorem, Tits [35, 36] and Weiss [39] derived the same conclusion as Feit and Higman, namely, that $n = 2$, 3, 4, 6 or 8.

As for quadrangles, the question of the existence of thick generalised n-gons (for $n \geq 3$) with s finite and t infinite is completely open. Of course, n must be even in such a geometry!

Exercises

1. Prove the assertions claimed to be "easy" in the text.
2. Construct infinite "free" generalised n-gons for any $n \geq 3$.

8.9 Some generalised hexagons

In this section, we use triality to construct a generalised hexagon called $G_2(F)$ over any field F. The construction is due to Tits. The name arises from the fact that the automorphism groups of these hexagons are the Chevalley groups of type G_2, as constructed by Chevalley from the simple Lie algebra G_2 over the complex numbers.

We begin with the triality quadric $\mathcal{Q}$. Let $\mathbf{v}$ be a non-singular vector. Then $\mathbf{v}^\perp \cap \mathcal{Q}$ is a rank 3 quadric. Its maximal t.s. subspaces are planes, and each lies in a unique solid of each family on $\mathcal{Q}$. Conversely, a solid on $\mathcal{Q}$ meets $\mathbf{v}^\perp$ in a plane. Thus, fixing $\mathbf{v}$, there is are bijections between the two families of solids and the set of planes on $\mathcal{Q}' = \mathcal{Q} \cap \mathbf{v}^\perp$. On this set, we have the structure of the *dual polar space* induced by the quadric $\mathcal{Q}'$; in other words, the POINTs are the planes on this quadric, the LINES are the lines, and incidence is reversed inclusion. Call this geometry $\mathcal{G}$.

Applying triality, we obtain a representation of $\mathcal{G}$ using all the points and some of the lines of $\mathcal{Q}$.

Now we take a non-singular vector, which may as well be the same as the vector $\mathbf{v}$ already used. (Since we have applied triality, there is no connection.)

The geometry $\mathcal{H}$ consists of those points and lines of $\mathcal{G}$ which lie in $\mathbf{v}^\perp$. Thus, it consists of all the points, and some of the lines, of the quadric $\mathcal{Q}'$.

Theorem 8.14 *$\mathcal{H}$ is a generalised hexagon.*

Proof First we observe some properties of the geometry $\mathcal{G}$, whose points and lines correspond to planes and lines on the quadric $\mathcal{Q}'$. The distance between two points is equal to the codimension of their intersection. If two planes of $\mathcal{Q}'$ meet non-trivially, then the corresponding solids of $\mathcal{Q}$ (in the same family) meet in a line, and so (applying triality) the points are perpendicular. Hence:

(a) Points of $\mathcal{G}$ lie at distance 1 or 2 if and only if they are perpendicular.

Let x, y, z, w be four points of $\mathcal{G}$ forming a 4-cycle. These points are pairwise perpendicular (by (a)), and so they span a t.s. solid S. We prove:

(b) The geometry induced on S by $\mathcal{G}$ is a symplectic GQ.

Keep in mind the following transformations:

solid S
$\to$ point p (by triality)
$\to$ quadric $\overline{\mathcal{Q}}$ in $p^\perp/p$ (residue of p)
$\to$ $\mathrm{PG}(3,F)$ (Klein correspondence).

Now points of S become solids of one family containing p, then planes of one family in $\overline{\mathcal{Q}}$, then points in $\mathrm{PG}(3,F)$; so we can identify the two ends of this chain.

Lines of $\mathcal{G}$ in S become lines through p perpendicular to $\mathbf{v}$, then points of $\overline{\mathcal{Q}}$ perpendicular to $\langle\overline{\mathbf{v}}\rangle = \langle \mathbf{v}, p\rangle/p$, then t.i. lines of a symplectic GQ, by the correspondence described in Section 8.3. Thus (b) is proved.

A property of $\mathcal{G}$ established in Proposition 7.9 is:

(c) If x is a point and L a line, then there is a unique point of L nearest to x.

We now turn our attention to $\mathcal{H}$, and observe first:

(d) Distances in $\mathcal{H}$ are the same as in $\mathcal{G}$.

For clearly distances in $\mathcal{H}$ are at least as great as those in $\mathcal{G}$, and two points of $\mathcal{H}$ at distance 1 (i.e., collinear) in $\mathcal{G}$ are collinear in $\mathcal{H}$.

Suppose that $x, y \in \mathcal{H}$ lie at distance 2 in $\mathcal{G}$. They are joined by more than one path of length 2 there, hence lie in a solid S carrying a symplectic GQ, as in (b). The points of $\mathcal{H}$ in S are those of $S \cap \mathbf{v}^{\perp}$, a plane on which the induced substructure is a plane pencil of lines of $\mathcal{H}$. Hence x and y lie at distance 2 in $\mathcal{H}$.

Finally, let $x, y \in \mathcal{H}$ lie at distance 3 in $\mathcal{G}$. Take a line L of $\mathcal{H}$ through y; there is a point z of $\mathcal{G}$ (and hence of $\mathcal{H}$) on L at distance 2 from x (by (c)). So x and y lie at distance 3 in $\mathcal{H}$.

In particular, property (c) holds also in $\mathcal{H}$.

(e) For any point x of $\mathcal{H}$, the lines of $\mathcal{H}$ through x form a plane pencil.

For, by (a), the union of these lines lies in a t.s. subspace, hence they are coplanar; there are no triangles (by (c)), so this plane contains two points at distance 2; now the argument for (d) applies.

Finally:

(f) $\mathcal{H}$ is a generalised hexagon.

We know it has diameter 3, and (GP2) is clearly true. A circuit of length less than 6 would be contained in a t.s. subspace, leading to a contradiction as in (d) and (e). (In fact, by (c), it is enough to exclude quadrangles.) ■

Cameron and Kantor [12] give a more elementary construction of this hexagon. Their construction, while producing the embedding in $\mathcal{Q}'$, depends only on properties of the group $\mathrm{PSL}(3, F)$. However, the proof that it works uses both counting arguments and arguments about finite groups; it is not obvious that it works in general, although the result remains true.

If F is a perfect field of characteristic 2 then, by Theorem 8.5, $\mathcal{Q}'$ is isomorphic to the symplectic polar space of rank 3; so $\mathcal{H}$ is embedded as all the points and some of the lines of $\mathrm{PG}(5, F)$.

Two further results will be mentioned without proof. First, if the field F has an automorphism of order 3, then the construction of $\mathcal{H}$ can be "twisted", much as can be done to the Klein correspondence to obtain the duality between orthogonal and unitary quadrangles (mentioned in Section 8.3), to produce another generalised hexagon, called ${}^3D_4(F)$. In the finite case, ${}^3D_4(q^3)$ has parameters $s = q^3$, $t = q$.

Second, there is a construction similar to that of Section 8.4. The generalised hexagon $G_2(F)$ is self-dual if F is a perfect field of characteristic 3, and is self-polar if F has an automorphism σ satisfying $\sigma^2 = 3$. In this case, the set of absolute points of the polarity is an ovoid, a set of pairwise non-collinear points meeting every line of $\mathcal{H}$, and the group of collineations commuting with the polarity has as a normal subgroup the *Ree group* ${}^2G_2(F)$, acting 2-transitively on the points of the ovoid.

Exercise

1. Show that the hexagon $\mathcal{H}$ has two disjoint planes E and F, each of which consists of pairwise non-collinear (but perpendicular) points. Show that each point of E is collinear (in $\mathcal{H}$) to the points of a line of F, and dually, so that E and F are naturally dual. Show that the points of $E \cup F$, and the lines of $\mathcal{H}$ joining their points, form a non-thick generalised hexagon which is the flag geometry of $\mathrm{PG}(2,F)$. (This is the starting point in the construction of Cameron and Kantor referred to in the text.)