5

Buekenhout geometries

Francis Buekenhout introduced an approach to geometry which has the advantages of being both general, and local (a geometry is studied *via* its residues of small rank). In this chapter, we introduce Buekenhout's geometries, and illustrate with projective spaces and related objects. Further examples will occur later (polar spaces).

5.1 Buekenhout geometries

So far, nothing has been said in general about what a "geometry" is. Projective and affine geometries have been defined as collections of subspaces, but even the structure carried by the set of subspaces was left a bit vague (except in Section 3.4, where we used the inclusion partial order to characterise generalised projective spaces as lattices). In this section, I will follow an approach due to Buekenhout (inspired by the early work of Tits on buildings).

Before giving the formal definition, let us remark that the subspaces or flats of a projective geometry are of various types (i.e., of various dimensions); may or may not be incident (two subspaces are incident if one contains the other); and are partially ordered by inclusion. To allow for duality, we do not want to take the partial order as basic; and, as we will see, the betweenness relation derived from it can be deduced from the type and incidence relations. So we regard type and incidence as basic.

A *geometry*, or *Buekenhout geometry*, then, has the following ingredients: a set X of *varieties*, a symmetric *incidence relation* I on X, a finite set Δ of *types*, and a *type map* $\tau : X \to \Delta$. We require the following axiom:

(B1) Two varieties of the same type are incident if and only if they are equal.

In other words, a geometry is a multipartite graph, where we have names for the multipartite blocks ("types") of the graph. We mostly use familiar geometric language for incidence; but sometimes, graph-theoretic terms like diameter and girth will be useful. But one graph-theoretic concept is vital; a geometry is *connected* if the graph of varieties and incidence is connected.

The *rank* of a geometry is the number of types.

A *flag* is a set of pairwise incident varieties. It follows from (B1) that the members of a flag have different types. A geometry satisfies the *transversality condition* if the following strengthening of (B1) holds:

(B2) (a) Every flag is contained in a maximal flag.

(b) Every maximal flag contains one variety of each type.

All geometries here will satisfy transversality.

Let F be a flag in a geometry G. The *residue* $G_F = R(F)$ of F is defined as follows: the set of varieties is

$$X_F = \{x \in X \setminus F : xIy \text{ for all } y \in F\};$$

the set of types is $\Delta_F = \Delta \setminus \tau(F)$; and incidence and the type map are the restrictions of those in G. It satisfies (B1) (resp. (B2)) if G does. The *type* of a flag or residue is its image under the type map, and the *cotype* is the complement of the type in Δ; so the type of G_F is the cotype of F. The *rank* and *corank* are the cardinalities of the type and cotype.

A transversal geometry is called *thick* (resp. *firm thin*) if every flag of corank 1 is contained in at least three (resp. at least two, exactly two) maximal flags.

A property holds *residually* in a geometry if it holds in every residue of rank at least 2. (Residues of rank 1 are sets without structure.) In particular, all geometries of interest are *residually connected*; in effect, we assume residual connectedness as an axiom:

(B3) All residues of rank at least 2 are connected.

The next result illustrates this concept.

Proposition 5.1 *Let G be a residually connected transversal geometry, and let x and y be varieties of X, and i and j distinct types. Then there is a path from x to y in which all varieties except possibly x and y have type i or j.*

Proof The proof is by induction on the rank. For rank 2, residual connectedness is just connectedness, and the result holds by definition. So assume the result for all geometries of smaller rank than G.

We show first that a two-step path whose middle vertex is not of type i or j can be replaced by a path of the type required. So let xzy be a path of length 2. Then x and y lie in the residue of z; so the assertion follows from the inductive hypothesis.

Now this construction reduces by one the number of interior vertices not of type i or j on a path with specified endpoints. Repeating it as often as necessary gives the result. ■

The heart of Buekenhout's idea is that "local" conditions on (or axiomatisations of) a geometry are really conditions about residues of small rank. This motivates the following definition of a diagram.

Let Δ be a finite set. Assume that, for any distinct $i,j \in \Delta$, a class $\mathcal{G}_{ij}$ of geometries of rank 2 is given, whose two types of varieties are called "points" and "blocks". Suppose that the geometries in $\mathcal{G}_{ji}$ are the duals of those in $\mathcal{G}_{ij}$. The set Δ equipped with these collections of geometries is called a *diagram*. It is represented pictorially by taking a "node" for each element of Δ, with an "edge" between each pair of nodes, the edge from i to j being adorned or labelled with some symbol for the class $\mathcal{G}_{ij}$. We will see examples later.

A geometry G belongs to the diagram $(\Delta, (\mathcal{G}_{ij} : i,j \in \Delta))$ if Δ is the set of types of G and, for all distinct $i,j \in \Delta$, and all residues G_F in G with rank 2 and type $\{i,j\}$, G_F is isomorphic to a member of $\mathcal{G}_{ij}$ (where we take points and blocks in G_F to be varieties of types i and j respectively).

In order to illustrate this idea, we need to define some classes of rank 2 geometries to use in diagrams. Some of these we have met already; but the most important is the most trivial: A *digon* is a rank 2 geometry (having at least two points and at least two blocks) in which any point and block are incident; in other words, a complete bipartite graph containing a cycle. By abuse of notation, the "labelled edge" used to represent digons is the absence of an edge! This is done in part because most of the rank 2 residues of our geometries will be digons, and this convention leads to uncluttered pictorial representations of diagrams.

A *partial linear space* is a rank 2 geometry in which two points lie on at most one line (and dually, two lines meet in at most one point). It is represented by an edge with the label Π, thus:

$$\overset{\Pi}{\circ\!\!-\!\!-\!\!-\!\!\circ}\,.$$

We already met the concepts *linear space* and *generalised projective plane*: they are partial linear spaces in which the first, resp. both, occurrences of "at most" are replaced by "exactly". They are represented by edges with label L and without any label, respectively. (Conveniently, the labels for the self-dual concepts of "partial linear space" and "generalised projective plane" coincide with their mirror-images, while the label for "linear space" does not.) Note that a projective plane is a thick generalised projective plane. Another specialisation of linear spaces, a "circle" or "complete graph", has all lines of cardinality 2; it is denoted by an edge with label c.

Now we can give an example:

Proposition 5.2 *A projective geometry of dimension n has the diagram*

Proof Transversality and residual connectivity are straightforward to check. We verify the rank 2 residues. Take the types to be the dimensions $0,1,\dots,n-1$, and let F be a flag of cotype $\{i,j\}$, where $i<j$.

Case 1: $j=i+1$. Then F has the form

$$U_0<U_1<\dots<U_{i-1}<U_{i+2}<\dots<U_{n-1}.$$

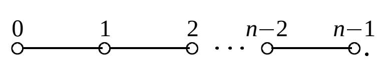
Its residue consists of all subspaces of dimension i or $i+1$ between U_{i-1} and U_{i+2}; this is clearly the projective plane based on the rank 3 vector space U_{i+2}/U_{i-1}.

Case 2: $j>i+1$. Now the flag F looks like

$$U_0<\dots<U_{i-1}<U_{i+1}<\dots<U_{j-1}<U_{j+1}<\dots<U_{n-1}.$$

Its residue consists of all subspaces lying either between U_{i-1} and U_{i+1}, or between U_{j-1} and U_{j+1}. Any subspace X of the first type is incident with any subspace Y of the second, since $X<U_{i+1}\le U_{j-1}<Y$. So the residue is a digon. ■

In diagrams, it is convenient to label the nodes with the corresponding elements of Δ. For example, in the case of a projective geometry of dimension n, we take the labels to be the dimensions of varieties represented by the nodes, thus:



I will use the convention that labels are placed above the nodes where possible. This reserves the space below the nodes for another use, as follows.

A transversal geometry is said to have *orders*, or *parameters*, if there are numbers s_i (for $i \in \Delta$) with the property that any flag of cotype i is contained in exactly $s_i + 1$ maximal flags. If so, these numbers s_i are the orders (or parameters). Now, if G is a geometry with orders, then G is thick/firm/thin respectively if and only if all orders are $> 1/\geq 1/= 1$ respectively. We will write the orders beneath the nodes, where appropriate. Note that a projective plane of order n (as defined earlier) has orders n,n (in the present terminology). Thus, the geometry $\mathrm{PG}(n,q)$ has diagram

$$\begin{array}{ccccccccc} 0 & & 1 & & 2 & & n-2 & & n-1 \\ \circ & \text{———} & \circ & \text{———} & \circ & \cdots & \circ & \text{———} & \circ . \\ q & & q & & q & & q & & q \end{array}$$

We conclude this section with some general results about Buekenhout geometries. These results depend on our convention that a non-edge symbolises a digon.

Proposition 5.3 *Let the diagram Δ be the disjoint union of Δ_1 and Δ_2, with no edges between these sets. Then a variety with type in Δ_1 and one with type in Δ_2 are incident.*

Proof We use induction on the rank. For rank 2, Δ is the diagram of a digon, and the result is true by definition. So assume that $|\Delta| > 2$, and (without loss of generality) that $|\Delta_1| > 1$.

Let X_i be the set of varieties with type in Δ_i, for $i = 1,2$. By the inductive hypothesis, if $x,y \in X_1$ with xIy, then $R(x) \cap X_2 = R(y) \cap X_2$. (Considering $R(x)$, we see that every variety in $R(x) \cap X_2$ is incident with y, so the left-hand set is contained in the right-hand set. Reversing the rôles of x and y establishes the result.) Now by connectedness, $R(x) \cap X_2$ is independent of $x \in X_1$. (Note that Proposition 5.1 is being used here.) But this set must be X_2, since every variety in X_2 is incident with some variety in X_1. ■

A diagram is *linear* if the “non-digon” edges form a simple path, as in the diagram for projective spaces in Proposition 5.3 above.

Suppose that one particular type in a geometry is selected, and varieties of that type are called points. Then the *shadow*, or *point-shadow*, of a variety x is the set $\mathrm{Sh}(x)$ of varieties incident with x. Sometimes we write $\mathrm{Sh}_0(x)$, where 0 is the type of a point. In a geometry with a linear diagram, the convention is that points are varieties of the left-most type.

Corollary 5.4 *In a linear diagram, if xIy, and the type of y is further to the right than that of x, then* $\mathrm{Sh}(x) \subseteq \mathrm{Sh}(y)$.

Proof $R(x)$ has disconnected diagram, with points and the type of y in different components; so, by Proposition 5.2, every point in $R(x)$ is incident with y. ■

Exercises

1. (a) Construct a geometry which is connected but not residually connected.

(b) Show that, if G has any of the following properties, then so does any residue of G of rank at least 2: residually connected, transversal, thick, firm, thin.

2. Show that any generalised projective geometry belongs to the diagram

$$\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\circ \cdots \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\circ.$$

3. (a) A *chamber* of a transversal geometry G is a maximal flag. Let $\mathcal{F}$ be the set of chambers of the geometry G. Form a graph with vertex set $\mathcal{F}$ by joining two chambers which coincide in all but one variety. G is said to be *chamber-connected* if this graph is connected. Prove that a residually connected geometry is chamber-connected, and a chamber-connected geometry is connected.

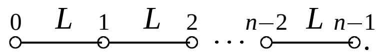
(b) Consider the 3-dimensional affine space $\mathrm{AG}(3,F)$ over the field F. Take three types of varieties: points (type 0), lines (type 1), and parallel classes of planes (type 2). Incidence between points and lines is as usual; a line L and a parallel class C of planes are incident if L lies in some plane of C; and any variety of type 0 is incident with any variety of type 2. Show that this geometry is chamber-connected but not residually connected.

(c) Let V be a six-dimensional vector space over a field F, with a basis $\{e_1,e_2,e_3, f_1, f_2, f_3\}$. Let G be the additive group of V, and let H_1, H_2, H_3 be the additive groups of the three subspaces $\langle e_2,e_3, f_1\rangle$, $\langle e_3,e_1, f_2\rangle$, and $\langle e_1,e_2, f_3\rangle$. Form the *coset geometry* $\mathcal{G}(G,(H_1,H_2,H_3))$: its vaarieties of type i are the cosets of H_i in G, and two varieties are incident if and only if the corresponding cosets have non-empty intersection. Show that this geometry is connected but not chamber-connected.

5.2 Some special diagrams

In this section, we first consider geometries with linear diagram in which all strokes are linear spaces; then we specialise some or all of these linear spaces to projective or affine planes. We will see that the axiomatisations of projective and affine spaces can be expressed very simply in this formalism.

Theorem 5.5 *Let G be a geometry with diagram*



Let varieties of type 0 and 1 be points and lines.

(a) The points and shadows of lines form a linear space $\mathcal{L}$.

(b) The shadow of any variety is a subspace of $\mathcal{L}$.

(c) $\mathrm{Sh}_0(x) \subseteq \mathrm{Sh}_0(y)$ *if and only if x is incident with y.*

(d) If x is a variety and p a point not incident with x, then there is a unique variety y incident with x and p such that $\tau(y) = \tau(x) + 1$.

Proof (a) We show that two points lie on at least one line by induction on the rank. There is a path between any two points using only points and lines, by Proposition 5.2; so it suffices to show that any such path of length greater than 2 can be shortened. So assume $pILIqIMIr$, where p, q, r are points and L, M lines. By the induction hypothesis, the POINTs L and M of $R(q)$ lie in a LINE Π, a plane of G incident with L and M. By Corollary 5.4, p and q are incident with Π. Since Π is a linear space, there is a line through p and q. (The convention of using capitals for varieties in $R(q)$ is used here.)

Now suppose that two lines L and M contain the two points p and q. Considering $R(p)$, we find a plane Π incident with L and M and hence with p and q. But Π is a linear space, so $L = M$.

(b) Let y be any variety, and $p, q \in \mathrm{Sh}_0(y)$. Since points and lines incident with y form a linear space by (a), there is a line incident with p, q and y. This must be the unique line incident with p and q; and, by Corollary 5.4, all its points are incident with y and so are in $\mathrm{Sh}_0(y)$.

(c) The reverse implication is Corollary 5.4. So suppose that $\mathrm{Sh}_0(x) \subseteq \mathrm{Sh}_0(y)$. Take $p \in \mathrm{Sh}_0(x)$. Then, in $R(p)$, we have $\mathrm{Sh}_1(x) \subseteq \mathrm{Sh}_1(y)$ (since these shadows are linear subspaces) , and so xIy by induction. (The base case of the induction, where x is a line, is covered by (b).)

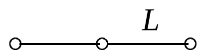
(d) This is clear if x is a point. Otherwise, choose $q \in \mathrm{Sh}_0(x)$, and apply induction in $R(q)$ (replacing p by the line pq). ∎

Theorem 5.6 *A geometry with diagram*

is a generalised projective space (of finite dimension).

Proof By Theorem 5.5(d), a potential Veblen configuration lies in a plane; since planes are projective, Veblen's axiom holds. It remains to show that every linear subspace is the shadow of some variety; this follows easily by induction. ■

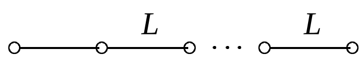
Theorem 5.7 *A geometry with diagram*



consists of the points, lines and planes of a (possibly infinite-dimensional) generalised projective space.

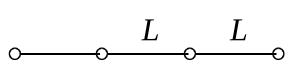
Proof Veblen's axiom is verified as in Theorem 5.6. It is clear that every point, line or plane corresponds to a variety. ■

Remark. Consider geometries with the diagram



By the argument for Theorem 5.7, we have all the points, lines and planes, and some higher-dimensional varieties, of a generalised projective space. Examples arise by taking all the flats of dimension at most $r-1$, where r is the rank. However, there are other examples. A simple case, with $r=4$, can be constructed as follows.

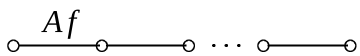
Let $\mathcal{P}$ be a projective space of countable dimension over a finite field F. Enumerate the 3-dimensional and 4-dimensional subspaces in lists $T_0, T_1, \ldots$ and $F_0, F_1, \ldots$. Now construct a set $\mathcal{F}$ of 4-dimensional subspaces in stages as follows. At the n^{th} stage, if T_n is already contained in a member of $\mathcal{F}$, do nothing. Otherwise, of the infinitely many subspaces F_j which contain T_n, only finitely many are excluded because they contain any T_m with $m<n$; let F_i be the one with smallest index which is not excluded, and adjoin it to $\mathcal{F}$. At the conclusion, any 3-dimensional subspace is contained in a unique member of $\mathcal{F}$. Then the points, lines, planes, and subspaces in $\mathcal{F}$ form a geometry with the diagram



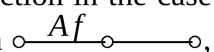
where the first L denotes the points and lines in 3-dimensional projective space over F.

Now we turn to affine spaces, where similar results hold. The label Af on a stroke will denote the class of affine planes.

Theorem 5.8 *A geometry with diagram*



is an affine space of finite dimension.

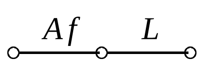
Proof It is a linear space whose planes are affine (that is, satisfying condition (AS1) of Section 11). We must show that parallelism is transitive. So suppose that $L_1 \| L_2 \| L_3$, but $L_1 \not\| L_3$. Then all three lines lie in a subspace of dimension 3; so it is enough to deduce a contradiction in the case of geometries of rank 3. Note that, for a geometry with diagram 

, two planes which have a common point must meet in a line.

Let Π_1 be the plane through L_1 and L_2, and Π_2 the plane through p and L_3, where p is a point of L_1. Then Π_1 and Π_2 both contain p, so they meet in a line $M \neq L_1$. Then M is not parallel to L_2, so meets it in a point q, But then Π_2 contains L_3 and q, hence L_2, and so is equal to Π_1, a contradiction.

The fact that all linear subspaces are shadows of varieties is proved as in Theorem 5.6. ■

Theorem 5.9 *A geometry with diagram*

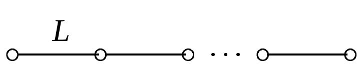


in which some line has more than three points, consists of the points, lines and planes of a (possibly infinite-dimensional) affine space.

The proof is as for Theorem 5.7, using Buekenhout's Theorem 3.10. ■

Exercises

1. Consider a geometry of rank n with diagram



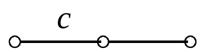
in which all lines have the same finite cardinality k, and all the projective planes have the same finite order q.

(a) If $n \geq 4$, prove that the geometry is either projective ($q = k - 1$) or affine ($q = k$).

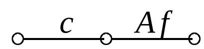
(b) If $n = 3$, prove that $q = k - 1, k, k^2$ or $k(k^2 + 1)$.

(This result is due to Doyen and Hubaut [16]).

2. Construct an infinite "free-like" geometry with diagram



(Ensure that three points lie in a unique plane, while two planes meet in two points.)

3. (a) Show that an inversive plane belongs to the diagram 

. What are the varieties?

(b) Show how to construct a geometry with diagram

(n nodes) from an ovoid in $\mathrm{PG}(n, F)$ (see Section 4.4).