2

Projective planes

Projective and affine planes are more than just spaces of smallest (non-trivial) dimension: as we will see, they are truly exceptional, and also they play a crucial rôle in the coordinatisation of arbitrary spaces.

2.1 Projective planes

We have seen in Sections 1.2 and 1.3 that, for any field F, the geometry $\mathrm{PG}(2,F)$ has the following properties:

(PP1) Any two points lie on exactly one line.

(PP2) Any two lines meet in exactly one point.

(PP3) There exist four points, no three of which are collinear.

I will now use the term *projective plane* in a more general sense, to refer to any structure of points and lines which satisfies conditions (PP1)-(PP3) above.

In a projective plane, let p and L be a point and line which are not incident. The incidence defines a bijection between the points on L and the lines through p. By (PP3), given any two lines, there is a point incident with neither; so the two lines contain equally many points. Similarly, each point lies on the same number of lines; and these two constants are equal. The *order* of the plane is defined to be one less than this number. The order of $\mathrm{PG}(2,F)$ is equal to the cardinality of F. (We saw in the last section that a projective line over $\mathrm{GF}(q)$ has $\left[{2 \atop 1}\right]_q = q+1$ points; so $\mathrm{PG}(2,q)$ is a projective plane of order q. In the infinite case, the claim follows by simple cardinal arithmetic.)

Given a finite projective plane of order n, each of the $n+1$ lines through a point p contains n further points, with no duplications, and all points are accounted for in this way. So there are n^2+n+1 points, and the same number of lines. The points and lines form a 2-$(n^2+n+1, n+1, 1)$ design. The converse is also true (see Exercise 2).

Do there exist projective planes not of the form $\mathrm{PG}(2,F)$? The easiest such examples are infinite; I give two completely different ones below. Finite examples will appear later.

Example 1: *Free planes.* Start with any configuration of points and lines having the property that two points lie on at most one line (and dually), and satisfying (PP3). Perform the following construction. At odd-numbered stages, introduce a new line incident with each pair of points not already incident with a line. At even-numbered stages, act dually: add a new point incident with each pair of lines for which such a point doesn't yet exist. After countably many stages, a projective plane is obtained. For given any two points, there will be an earlier stage at which both are introduced; by the next stage, a unique line is incident with both; and no further line incident with both is added subsequently; so (PP1) holds. Dually, (PP2) holds. Finally, (PP3) is true initially and remains so. If we start with a configuration violating Desargues' Theorem (for example, the Desargues configuration with the line pqr "broken" into separate lines pq, qr, rp), then the resulting plane doesn't satisfy Desargues' Theorem, and so is not a $\mathrm{PG}(2,F)$.

Example 2: *Moulton planes.* Take the ordinary real affine plane. Imagine that the lower half-plane is a refracting medium which bends lines of positive slope so that the part below the axis has twice the slope of the part above, while lines with negative (or zero or infinite) slope are unaffected. This is an affine plane, and has a unique completion to a projective plane (see later). The resulting projective plane fails Desargues' theorem. To see this, draw a Desargues configuration in the ordinary plane in such a way that just one of its ten points lies below the axis, and just one line through this point has positive slope.

The first examples of finite planes in which Desargues' Theorem fails were constructed by Veblen and Wedderburn [38]. Many others have been found since, but all known examples have prime power order. The *Bruck–Ryser Theorem* [4] asserts that, if a projective plane of order n exists, where $n \equiv 1$ or 2 (mod 4), then n must be the sum of two squares. Thus, for example, there is no projective plane of order 6 or 14. This theorem gives no information about 10, 12, 15, 18, $\ldots$.

Recently, Lam, Swiercz and Thiel [21] showed by an extensive computation that there is no projective plane of order 10. The other values mentioned are undecided.

An *affine plane* is an incidence structure of points and lines satisfying the following conditions (in which two lines are called *parallel* if they are equal or disjoint):

(AP1) Two points lie on a unique line.

(AP2) Given a point p and line L, there is a unique line which contains p and is parallel to L.

(AP3) There exist three non-collinear points.

Remark. Axiom (AP2) for the real plane is an equivalent form of Euclid's "parallel postulate". It is called "Playfair's Axiom", although it was stated explicitly by Proclus.

Again it holds that $\mathrm{AG}(2,F)$ is an affine plane. More generally, if a line and all its points are removed from a projective plane, the result is an affine plane. (The removed points and line are said to be "at infinity". Two lines are parallel if and only if they contain the same point at infinity.

Conversely, let an affine plane be given, with point set $\mathcal{P}$ and line set $\mathcal{L}$. It follows from (AP2) that parallelism is an equivalence relation on $\mathcal{L}$. Let $\mathcal{Q}$ be the set of equivalence classes. For each line $L \in \mathcal{L}$, let $L^+ = L \cup \{Q\}$, where Q is the parallel class containing L. Then the structure with point set $\mathcal{P} \cup \mathcal{Q}$, and line set $\{L^+ : L \in \mathcal{L}\} \cup \{\mathcal{Q}\}$, is a projective plane. Choosing $\mathcal{Q}$ as the line at infinity, we recover the original affine plane.

We will have more to say about affine planes in Section 3.5.

Exercises

1. Show that a structure which satisfies (PP1) and (PP2) but not (PP3) must be of one of the following types:

(a) There is a line incident with all points. Any further line is a singleton, repeated an arbitrary number of times.

(b) There is a line incident with all points except one. The remaining lines all contain two points, the omitted point and one of the others.

2. Show that a 2-$(n^2+n+1, n+1, 1)$ design (with $n > 1$) is a projective plane of order n.

3. Show that, in a finite affine plane, there is an integer $n > 1$ such that

- every line has n points;
- every point lies on $n+1$ lines;
- there are n^2 points;
- there are $n+1$ parallel classes with n lines in each.

(The number n is the *order* of the affine plane.)

4. (The *Friendship Theorem*.) In a finite society, any two individuals have a unique common friend. Prove that there exists someone who is everyone else's friend.

[Let X be the set of individuals, $\mathcal{L} = \{F(x) : x \in X\}$, where $F(x)$ is the set of friends of X. Prove that, in any counterexample to the theorem, $(X, \mathcal{L})$ is a projective plane, of order n, say.

Now let A be the real matrix of order n^2+n+1, with (x,y) entry 1 if x and y are friends, 0 otherwise. Prove that

$$A^2 = nI + J,$$

where I is the identity matrix and J the all-1 matrix. Hence show that the real symmetric matrix A has eigenvalues $n+1$ (with multiplicity 1) and $\pm\sqrt{n}$. Using the fact that A has trace 0, calculate the multiplicity of the eigenvalue $\sqrt{n}$, and hence show that $n = 1$.]

5. Show that any Desargues configuration in a free projective plane must lie within the starting configuration. [Hint: Suppose not, and consider the last point or line to be added.]

2.2 Desarguesian and Pappian planes

It is no coincidence that we distinguished the free and Moulton planes from $\mathrm{PG}(2,F)$s in the last section by the failure of Desargues' Theorem.

Theorem 2.1 *A projective plane is isomorphic to* $\mathrm{PG}(2,F)$ *for some* F *if and only if it satisfies Desargues' Theorem.*

I do not propose to give a detailed proof of this important result; but some comments on the proof are in order.

We saw in Section 1.3 that, in $\mathrm{PG}(2,F)$, the field operations (addition and multiplication) can be defined geometrically, once a set of four points with no three collinear has been chosen. By (PP3), such a set of points exists in any projective plane. So it is possible to define two binary operations on a set consisting of a line with a point removed, and to coordinatise the plane with this algebraic object. Now it is obvious that any field axiom translates into a certain "configuration theorem", so that the plane is a $\mathrm{PG}(2,F)$ if and only if all these "configuration theorems" hold. What is not obvious, and quite remarkable, is that all these "configuration theorems" follow from Desargues' Theorem.

Another method, more difficult in principle but much easier in detail, exploits the relation between Desargues' Theorem and collineations.

Let p be a point and L a line. A *central collineation* with centre p and axis L is a collineation fixing every point on L and every line through p. It is called an *elation* if p is on L, a *homology* otherwise. The central collineations with centre p and axis L form a group. The plane is said to be (p,L)-*transitive* if this group permutes transitively the set $M \setminus \{p, L\cap M\}$ for any line $M \neq L$ on p (or, equivalently, the set of lines on q different from L and pq, where $q \neq p$ is a point of L).

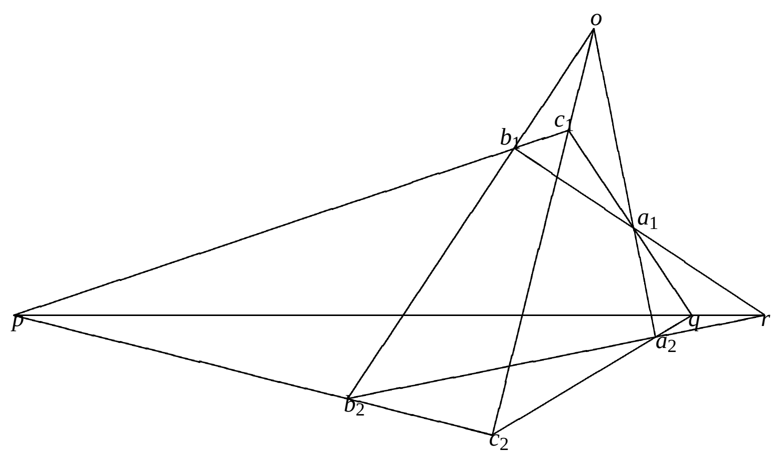

Figure 2.1: The Desargues configuration

Theorem 2.2 *A projective plane satisfies Desargues' Theorem if and only if it is (p,L)-transitive for all points p and lines L.*

Proof Let us take another look at the Desargues configuration (Fig. 2.1). It is clear that any central configuration with centre o and axis L which carries a_1 to a_2 is completely determined at every point b_1 not on M. (The line a_1a_2 meets L at a fixed point r and is mapped to b_1b_2; so b_2 is the intersection of ra_2 and ob_1.) Now, if we replace M with another line M' through o, we get another determination of the action of the collineation. It is easy to see that the condition that these two specifications agree is precisely Desargues' Theorem.

The proof shows a little more. Once the action of the central collineation on one point of $M \setminus \{o, L \cap M\}$ is known, the collineation is completely determined. So, if Desargues' Theorem holds, then these groups of central collineations act sharply transitively on the relevant set.

Now the additive and multiplicative structures of the field turn up as groups of elations and homologies respectively with fixed centre and axis. We see immediately that these structures are both groups. More of the axioms are easily deduced too. For example, let L be a line, and consider all elations with axis L (and arbitrary centre on L). This set is a group G. For each point p on L, the elations with centre p form a normal subgroup. These normal subgroups partition the non-identity elements of G, since a non-identity elation has at most one centre. But a group having such a partition is abelian (see Exercise 2). So addition is commutative.

In view of this theorem, projective planes over skew fields are called *Desarguesian planes*.

There is much more to be said about the relationships among configuration theorems, coordinatisation, and central collineations. I refer to Dembowski's book for some of these. One such relation is of particular importance.

Pappus' Theorem is the assertion that, if alternate vertices of a hexagon are collinear (that is, the first, third and fifth, and also the second, fourth and sixth), then also the three points of intersection of opposite edges are collinear. See Fig. 2.2.

Theorem 2.3 *A projective plane satisfies Pappus' Theorem if and only if it is isomorphic to* $\mathrm{PG}(2, F)$ *for some commutative field* F.

Proof The proof involves two steps. First, a purely geometric argument shows that Pappus' Theorem implies Desargues'. This is shown in Fig. 2.3. This figure shows a potential Desargues configuration, in which the required collinearity is shown by three applications of Pappus' Theorem. The proof requires four new

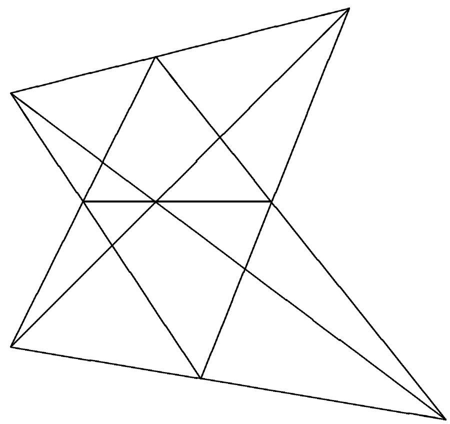

Figure 2.2: Pappus' Theorem

points, $s = a_1b_1 \cap a_2c_2$, $t = b_1c_1 \cap os$, $u = b_1c_2 \cap oa_1$, and $v = b_2c_2 \cap os$. Now Pappus' Theorem, applied to the hexagon $osc_2b_1c_1a_1$, shows that q, u, t are collinear; applied to $osb_1c_2b_2a_2$, shows that r, u, v are collinear; and applied to b_1tuvc_2s (using the two collinearities just established), shows that p, q, r are collinear. The derived collinearities are shown as dotted lines in the figure. (Note that the figure shows only the generic case of Desargues' Theorem; it is necessary to take care of the possible degeneracies as well.)

The second step involves the use of coordinates to show that, in a Desarguesian plane, Pappus' Theorem is equivalent to the commutativity of multiplication. (See Exercise 3.)

In view of this, projective planes over commutative fields are called *Pappian planes*.

Remark. It follows from Theorems 2.1 and 2.3 and Wedderburn's Theorem 1.1 that, in a finite projective plane, Desargues' Theorem implies Pappus'. No geometric proof of this implication is known.

A similar treatment of affine planes is possible.

Exercises

1. (a) Show that a collineation which has a centre has an axis, and *vice versa*.

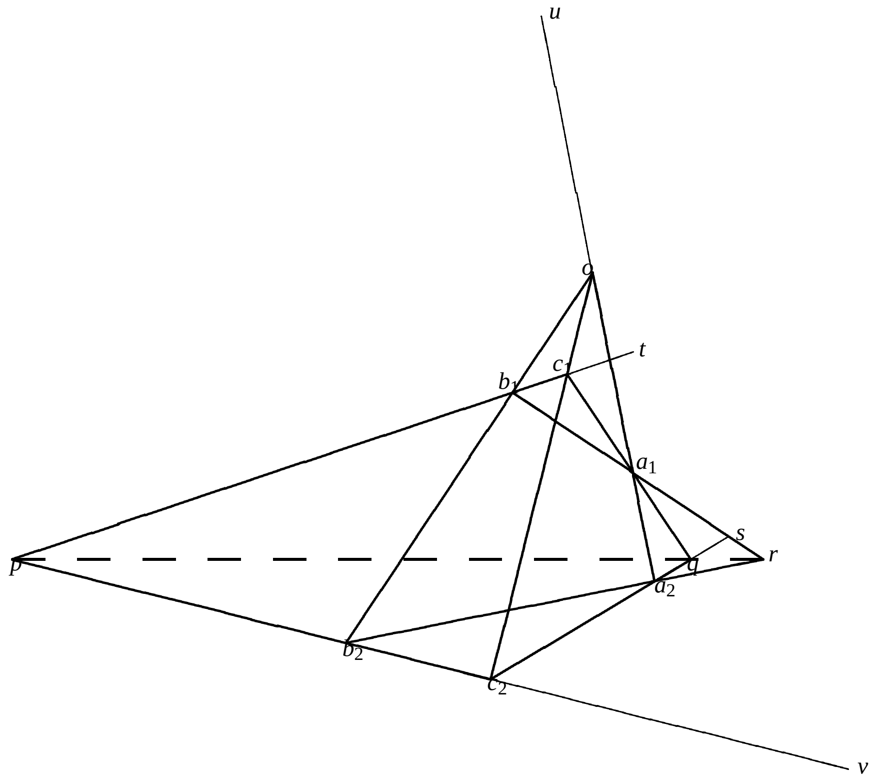

Figure 2.3: Pappus implies Desargues

(b) Show that a collineation cannot have more than one centre.

2. The group G has a family of proper normal subgroups which partition the non-identity elements of G. Prove that G is abelian.

3. In $\mathrm{PG}(2,F)$, let the vertices of a hexagon be $(1,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,\alpha+1,1)$, $(1,1,0)$ and $(\beta,\beta(\alpha+1),1)$. Show that alternate vertices lie on the lines defined by the column vectors $(0,0,1)^\top$ and $(\alpha+1,-1,0)^\top$. Show that opposite sides meet in the points $(\alpha,0,-1)$, $(0,\beta\alpha,1)$ and $(1,\beta(\alpha+1),1)$. Show that the second and third of these lie on the line $(\beta,-1,\beta\alpha)^\top$, which also contains the first if and only if $\alpha\beta=\beta\alpha$.

2.3 Projectivities

Let $\Pi = (X, \mathcal{L})$ be a projective plane. Temporarily, let (L) be the set of points incident with L; and let (x) be the set of lines incident with x. If x is not incident with L, there is a natural bijection between (L) and (x): each point on L lies on a unique line through x. This bijection is called a *perspectivity*. By iterating perspectivities and their inverses, we get a bijection (called a *projectivity*) between any two sets (x) or (L). In particular, for any line L, we obtain a set $P(L)$ of projectivities from (L) to itself (or *self-projectivities*), and analogously a set $P(x)$ for any point x.

The sets $P(L)$ and $P(x)$ are actually groups of permutations of (L) or (x). (Any self-projectivity is the composition of a chain of perspectivities; the product of two self-projectivities corresponds to the concatenation of the chains, while the inverse corresponds to the chain in reverse order.) Moreover, these permutation groups are naturally isomorphic: if g is any projectivity from (L_1) to (L_2), say, then $g^{-1}P(L_1)g = P(L_2)$. So the group $P(L)$ of self-projectivities on a line is an invariant of the projective plane. It turns out that the structure of this group carries information about the plane which is closely related to concepts we have already seen.

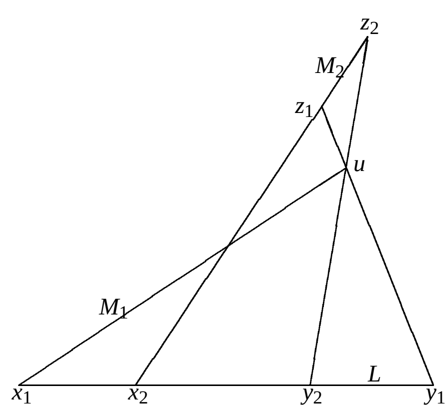

Figure 2.4: 3-transitivity

Proposition 2.4 *The permutation group $P(L)$ is 3-transitive.*

Proof It suffices to show that there is a projectivity fixing any two points $x_1, x_2 \in L$ and mapping any further point y_1 to any other point y_2. In general, we will use the notation "(L_1 to L_2 via p)" for the composite of the perspectivities $(L_1) \to (p)$ and $(p) \to (L_2)$. Let M_i be any other lines through x_i ($i = 1,2$), u a point on M_1, and $z_i \in M_2$ ($i = 1,2$) such that $y_i u z_i$ are collinear ($i = 1,2$). Then the product of (L to M_1 via z_1) and (M_1 to L via z_2) is the required projectivity (Fig. 2.4.)

A permutation group G is *sharply t-transitive* if, given any two t-tuples of distinct points, there is a unique element of G carrying the first to the second (in order). The main result about groups of projectivities is the following theorem:

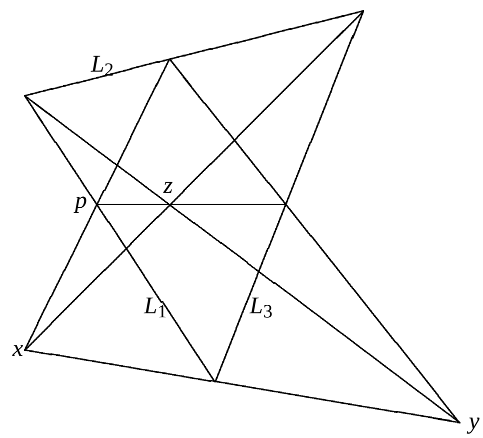

Figure 2.5: Composition of projectivities

Theorem 2.5 *The group $P(L)$ of projectivities on a projective plane Π is sharply 3-transitive if and only if Π is pappian.*

Proof We sketch the proof. The crucial step is the equivalence of Pappus' Theorem to the following assertion:

> Let L_1, L_2, L_3 be non-concurrent lines, and x and y two points such that the projectivity
>
> $$g = (L_1 \text{ to } L_2 \text{ via } x) \cdot (L_2 \text{ to } L_3 \text{ via } y)$$
>
> fixes $L_1 \cap L_3$. Then there is a point z such that the projectivity g is equal to (L_1 to L_3 via z).

The hypothesis is equivalent to the assertion that x, y and $L_1 \cap L_3$ are collinear. Now the point z is determined, and Pappus' Theorem is equivalent to the assertion that it maps a random point p of L_1 correctly. (Fig. 2.5 is just Pappus' Theorem.)

Now this assertion allows long chains of projectivities to be shortened, so that their action can be controlled.

The converse can be seen another way. By Theorem 2.3, we know that a Pappian plane is isomorphic to $\mathrm{PG}(2,F)$ for some commutative field F. Now it is easily checked that any self-projectivity on a line is induced by a linear fractional transformation (an element of $\mathrm{PGL}(2,F)$; and this group is sharply 3-transitive.

In the finite case, there are very few 3-transitive groups apart from the symmetric and alternating groups; and, for all known non-Pappian planes, the group of projectivities is indeed symmetric or alternating (though it is not known whether this is necessarily so). Both possibilities occur; so, at present, all that this provides us for non-Pappian finite planes is a single Boolean invariant.

In the infinite case, however, more interesting possibilities arise. If the plane has order α, then the group of projectivities has α generators, and so has order α; so it can never be the symmetric group (which has order 2^α). Barlotti [1] gave an example in which the stabiliser of any six points is the identity, and the stabiliser of any five points is a free group. On the other hand, Schleiermacher [25] showed that, if the stabiliser of any five points is trivial, then the stabiliser of any three points is trivial (and the plane is Pappian).

Further developments involve deeper relationships between projectivities, configuration theorems, and central collineations; the definition and study of projectivities in other incidence structures; and so on.